

A pedagogical presentation of a C^* –algebraic approach to quantum tomography

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Abstract. It is now well established that quantum tomography provides an alternative picture of quantum mechanics. It is common to introduce tomographic concepts starting with the Schrodinger-Dirac picture of quantum mechanics on Hilbert spaces. In this picture states are a primary concept and observables are derived from them. On the other hand, the Heisenberg picture, which has evolved in the C^* –algebraic approach to quantum mechanics, starts with the algebra of observables and introduce states as a derived concept. The equivalence between these two pictures amounts essentially, to the Gelfand-Naimark-Segal construction. In this construction, the abstract C^* –algebra is realized as an algebra of operators acting on a constructed Hilbert space. The representation one defines may be reducible or irreducible, but in either case it allows to identify a unitary group associated with the C^* –algebra by means of its invertible elements. In this picture both states and observables are appropriate functions on the group, it follows that also quantum tomograms are strictly related with appropriate functions (positive-type) on the group. In this paper we present, by means of very simple examples, the tomographic description emerging from the set of ideas connected with the C^* –algebra picture of quantum mechanics. In particular, the tomographic probability distributions are introduced for finite and compact groups and an autonomous criterion to recognize a given probability distribution as a tomogram of quantum state is formulated.

Key words C^* –algebras, finite groups, compact groups, quantum tomograms.

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1. Introduction

The problem of quantum state description was the subject of intensive investigations from the very beginning of quantum mechanics [1, 2, 3, 4, 5, 6, 7, 8]. The association of quantum states with quasi-distributions [4, 5, 6, 7] made the description of states in quantum mechanics similar to the description of classical particle states in classical statistical mechanics by means of probability distributions on phase space. However, the class of quasi-distributions introduced in quantum mechanics cannot contain all classical distribution functions because of the uncertainty relation [9, 10, 11].

In view of the uncertainty relations there cannot exist a joint probability distribution function for instance of two random position and momentum since they cannot be measured simultaneously. It is admissible to have a probability distribution function of only one of the two conjugate variables, for instance position.

The problem of discussing the position probability distribution together with the momentum probability distribution was discussed by Pauli [12]. Although this problem as formulated by Pauli found negative solution, it triggered investigations in this direction and it turned out that one can introduce a family of actual probability distributions of one random variable (position) called tomographic probability distributions or simply tomograms, these distributions provide a description of quantum states in complete analogy with the description of states in classical statistical mechanics [13], see also the recent review [14] and [15, 16, 17]. It is worthy to note that in [18, 19, 20, 21] the probability approach to describe the quantum states was discussed but the tomographic version of such description has appeared only as a result of thinking on experiments on homodyne detection of quantum photon states [22, 23] based on optical tomograms whose relation with the Wigner functions was found in [24, 25]. In these papers the tomograms, being measurable probability distributions, were considered as a technical tool to measure the photon quantum states identified with the Wigner functions. So, tomograms were not considered as primary objects providing an alternative picture of Quantum Mechanics.

In the papers [13] and [15, 16, 17] a new element in the tomographic approach to quantum mechanics appeared in the sense that the tomographic distribution itself is identified with the quantum state. In other words, knowing a quantum tomogram one can obtain all the quantities of quantum mechanics like the energy spectrum, quantum transition probabilities, quantum state evolution in the form of an equation for the probability distribution, etc. Thus the tomogram can be used as alternative to such primary concept of state as the notion of wave function or density operator (we also call it a density state).

According to the tomographic approach, for any density state (or wave function) one constructs the tomogram and *vice versa*, from any given tomogram one can reconstruct the quantum state density operator ρ . The density operator ρ has the properties: Hermiticity, i.e. $\rho^\dagger = \rho$, trace normalization $\text{Tr}[\rho] = 1$ and non-negativity, i.e. $\rho \geq 0$. The tomographic probability distribution provides the density operator by using the

inversion formulae that are available in explicit form for all kinds of tomograms like the optical one [26], symplectic tomogram [27, 28], spin tomogram [29], photon number tomogram [30] and center of mass tomogram [31]. The problem of measuring continuous position and momentum in connection with the tomographic description of quantum states was discussed in [32] and the discrete spin variables were considered in an analogous representation in [33].

If we consider from the very beginning the tomograms as conceptual primary objects associated with quantum states, the question arises for finding conditions to recognize whether a given probability distribution is a tomogram of a quantum state. The common answer to this question is that one has to use the inversion formula to obtain an operator $\tilde{\rho}$ and then to check if it has all the properties characterizing density operators. However this answer is unsatisfactory, because it requires to switch from the tomographic description to other pictures of quantum mechanics. Analogous problem was considered for Wigner functions and the criterion was formulated in [34] on the base of the so called Kastler-Loupas-Miracle Sole (KLM) conditions [35, 36, 37]. The connection of tomograms with Wigner functions could be used [38], but again it would be unsatisfactory. One needs autonomous criteria to answer the question.

In the present paper we provide self-contained conditions for a probability distribution to be a tomogram of a quantum state, i.e., a quantum tomogram. We will formulate such properties by using the Naimark method [39] and the Gelfand-Naimark-Segal (GNS) construction [40] to describe quantum states in terms of vectors of a suitable Hilbert space.

It is worthy to note that the tomographic approach can be formulated in the framework of a star-product scheme [41, 42, 43].

The strategy of our work is to find the connection between functions (which are diagonal matrix elements of a unitary representation of a group G) and quantum tomograms. Any such a function in [39] was shown to have properties of positivity, recalled in the following. Then, based on this property and in view of the connection with tomograms, we can establish the properties characterizing quantum tomograms among other probability distributions.

The paper is organized as follows. In sections 2 and 3 introductory remarks on C^* -algebras and a simple example are discussed. A concrete case of C^* -algebra, the group algebra, is discussed in section 4 for a finite group. The fundamental notion of positive-type group function is recalled in section 5. Section 6 is devoted to introduce the tomographic descriptions of quantum states based on irreducible representations of a finite group, *via* a positive group function. Section 6 is the core of this paper: its definitions and results, which are discussed with extreme detail in the case of the group of permutations of three points S_3 in the long section 7, are straightforwardly extended to the compact groups like $U(n)$ in section 8, after a *caveat* on the necessity of using the Gelfand-Zetlin bases [44, 45]. Also, the tomographic reconstruction formula provided in section 8 is evaluated in detail for the case of $SU(2)$. The Gelfand-Zetlin bases are discussed with some care in section 9. The paradigmatic case of $SU(3)$ illustrates the

previous results in section 10. In section 11 the important necessary and sufficient conditions for a given family of stochastic vectors to be a tomogram are formulated in terms of a suitable positive-type group function, both for finite and compact groups. An example based on S_3 illustrates the theory. Moreover, the possibility of checking the positivity of a compact group function *via* the restriction to a finite subgroup is analyzed. Finally, in section 12 some conclusions and perspectives are drawn.

2. Introductory remarks on C^* -algebras

It is known that the formulation of quantum mechanics stemming from Heisenberg picture is given by using a C^* -algebra formalism [46]. In this formalism from the very beginning one does not use neither a Hilbert space nor operators. Instead, it is used an associative algebra \mathcal{A} with identity E and a \star -involution, such that $(AB)^\star = B^\star A^\star$, plus an appropriate norm $\|\cdot\|$ to introduce a topology. The norm $\|\cdot\|$ satisfies the continuity requirement for the product $\|AB\| \leq \|A\| \|B\|$ and the compatibility condition $\|A^\star A\| = \|A\|^2$.

The observables of the theory are real (also called self-adjoint) elements: $A^\star = A$. States are normalized positive continuous linear functionals ρ on this algebra, this is, continuous linear maps such that $\rho(A^\star A) \geq 0$, and $\rho(E) = 1$, (replacing the trace property for density states). The mean value of an observable A in the state ρ , say $\langle A \rangle_\rho$, is just the real number $\rho(A)$, the evaluation of ρ on A .

Some elements of the algebra \mathcal{A} have an inverse. The elements U for which $U^\star = U^{-1}$ are called the unitary elements in the C^* -algebra, and they form a group \mathcal{U} .

Starting from a C^* -algebra \mathcal{A} , the Gelfand-Naimark-Segal (GNS) construction provides, given a fiducial state ρ , a Hilbert space carrying a \star -cyclic representation Π of the algebra, $\Pi(A^\star) = (\Pi(A))^\dagger$. In this way one gets density operators for states and Hermitian operators for observables of the usual formulation of quantum mechanics.

One of the aim of our work is to introduce the tomographic approach at the level of the C^* -algebra formulation of quantum mechanics and to relate it with standard formulation by means of the GNS construction.

The idea of a tomographic picture in a C^* -algebra is based on the possibility to represent an observable A , at least for group algebras based on compact groups as it will be discussed in the following, as real linear combination of projectors, this is in the form

$$A = \sum_{\alpha, k} \lambda_k^\alpha P_k^\alpha, \quad (1)$$

where λ_k^α are real numbers, the observables $P_k^\alpha = P_k^{\alpha\star}$ are such that $P_k^\alpha P_j^\beta = P_k^\alpha \delta_{\alpha\beta} \delta_{kj}$, and satisfying $\sum_{\alpha, k} P_k^\alpha = E$. It follows $AP_k^\alpha = \lambda_k^\alpha P_k^\alpha$.

The same kind of decomposition (1) for a $g \in \mathcal{U}$ gives $\lambda_k^\alpha = \exp(i\theta_k^\alpha)$, $\theta_k^\alpha \in \mathbb{R}$. Now for any state ρ , $\rho(P_k^\alpha) = \rho(P_k^{\alpha\star} P_k^\alpha) \geq 0$, so that we may interpret the formula $\rho(U) = \sum_{\alpha, k} \exp(i\theta_k^\alpha) \rho(P_k^\alpha)$ as the evaluation of the state ρ in U , providing the value of each random phase θ_k^α with probability $W_k^\alpha(\rho, U) := \rho(P_k^\alpha)$. In other words, we have

thus defined the tomographic probability $W_k^\alpha(\rho, U)$ of random index k for any given α , and write $\langle U \rangle_\rho =: \sum_{\alpha,k} \exp(i\theta_k^\alpha) W_k^\alpha(\rho, U)$.

We complete the construction by introducing the notion of the Naimark matrix $\mathcal{N}_{ij} = \rho(U_i^{-1}U_j)$, where i, j vary over any finite set of natural numbers. If it is positive semi-definite, that is $\sum_{i,j} \mathcal{N}_{ij} \bar{\xi}^i \xi^j \geq 0$ for all $\xi^i \in \mathbb{C}$, by definition, $\rho(U)$ is a positive-type function on the group \mathcal{U} . Finally, particular realizations of C^* -algebras as unitary irreducible representations of different groups provide corresponding standard definitions of tomography. The use of C^* -algebras constructed from groups makes possible explicit state reconstruction from its tomogram.

3. An introductory example

In this section we illustrate the notion of C^* -algebra by considering a simple finite dimensional example. Given three orthonormal vectors in a Hilbert space $|a_1\rangle, |a_2\rangle, |a_3\rangle$, let us consider the linear space \mathcal{A}_9 with nine base vectors organized in a table

$$\begin{bmatrix} |a_1\rangle\langle a_1| & |a_1\rangle\langle a_2| & |a_1\rangle\langle a_3| \\ |a_2\rangle\langle a_1| & |a_2\rangle\langle a_2| & |a_2\rangle\langle a_3| \\ |a_3\rangle\langle a_1| & |a_3\rangle\langle a_2| & |a_3\rangle\langle a_3| \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix}. \quad (2)$$

We define the table of products for the base vectors corresponding to the products of projectors

$$|a_j\rangle\langle a_m| |a_n\rangle\langle a_k| = \delta_{m,n} |a_j\rangle\langle a_k|. \quad (3)$$

The multiplication of the algebra elements is determined by multiplication of the basis vectors, whose multiplication table reads:

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \end{array} \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 \\ A_1 & A_2 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & A_5 & A_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_7 & A_8 & A_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_7 & A_8 & A_9 \end{bmatrix} \quad (4)$$

This multiplication table defines the structure constants of the algebra according to

$$A_j \cdot A_k = \gamma_{jk}^l A_l. \quad (5)$$

These structure constants satisfy the quadratic equations arising from associativity $\gamma_{jk}^m \gamma_{ml}^r = \gamma_{kl}^m \gamma_{jm}^r$. The association

$$A_j \Rightarrow (\gamma_j)_k^l := \gamma_{jk}^l \quad (6)$$

provides a realization of the algebra \mathcal{A}_9 in terms of 9×9 matrices.

Any element A of the C^* -algebra is defined as a complex linear combination:

$$A = \sum_{m=1}^9 c_m A_m = \sum_{j,k=1}^3 c_{jk} |a_j\rangle \langle a_k|. \quad (7)$$

The zero vector is given by $\{c_m = 0\}$ or equivalently $\{c_{jk} = 0\}$. The \star -involution is defined by:

$$(|a_j\rangle \langle a_k|)^\star = |a_k\rangle \langle a_j|, \quad (8)$$

what implies:

$$A_1^\star = A_1, \quad A_2^\star = A_4, \quad A_3^\star = A_7, \quad A_5^\star = A_5, \quad A_6^\star = A_8, \quad A_9^\star = A_9. \quad (9)$$

In general, for a linear combination (7), we define:

$$A^\star = \sum_{m=1}^9 c_m^\star A_m^\star = \sum_{j,k=1}^3 c_{jk}^\star |a_k\rangle \langle a_j|. \quad (10)$$

In view of the product table above, the unity element E satisfying $AE = EA = A$, for any A , is defined as

$$E = A_1 + A_5 + A_9. \quad (11)$$

As for the norm we may use the usual operator matrix norm.

The unitary elements U in the algebra \mathcal{A}_9 , satisfying $U^\star U = UU^\star = E$, are the elements such that $U^\star = U^{-1}$. The inverse element A^{-1} of A given by Eq.(7) exists if and only if (iff) $\det(c_{jk}) \neq 0$. Moreover, the element is unitary iff the representative matrix (c_{jk}) is unitary, i.e.:

$$\sum_{j=1}^3 c_{hj} c_{kj}^\star = \delta_{h,k}. \quad (12)$$

So, the unitary group \mathcal{U} is isomorphic to the group $U(3)$.

To any state ρ we can associated a vector A_ρ in the C^* -algebra \mathcal{A}_9 by means of the formula:

$$A_\rho = \sum_{m=1}^9 \rho(A_m^\star) A_m = \sum_{j,k=1}^3 \rho(|a_k\rangle \langle a_j|) |a_j\rangle \langle a_k|. \quad (13)$$

The vector A_ρ is real, $A_\rho^\star = A_\rho$, because states must be hermitian functionals $\rho(A^\star) = [\rho(A)]^\star$. Moreover, the hermitian matrix $(\rho_{jk}) = \rho(|a_k\rangle \langle a_j|)$ has trace one because $\rho(E) = \rho(A_1 + A_3 + A_5) = \rho(\sum_{k=1}^3 |a_k\rangle \langle a_k|) = \sum_{k=1}^3 \rho_{kk}$. The matrix ρ_{jk} is definite positive because $\rho(A^\star A) = \sum_{j,m,k,n=1}^9 c_{jm}^\star c_{kn} \rho(|a_m\rangle \langle a_j| |a_k\rangle \langle a_n|) = \sum_{k,m,n=1}^9 c_{km}^\star c_{kn} \rho_{mn}$ for arbitrary c_{km} . The positive definite property of the matrix ρ_{kj} can also be seen by considering the orbits of the transitive action of the unitary group \mathcal{U} through the vectors. Thus if we consider a diagonal state ρ^0 , i.e., a state such that $\rho(|a_n\rangle \langle a_n|) = \rho_n^0 \geq 0$, $n = 1, 2, 3$, and zero otherwise, we have:

$$A_{\rho^0} = \sum_{n=1}^3 \rho_n^0 (|a_n\rangle \langle a_n|) |a_n\rangle \langle a_n| = \sum_{n=1}^3 \rho_n^0 |a_n\rangle \langle a_n|, \quad (14)$$

Now, if U is a unitary element of \mathcal{A}_9 with representative matrix u , then we will denote by ρ_U the state $\rho_U(A) = \rho(U^*AU)$ which give all the state vectors A_ρ as:

$$\begin{aligned} A_\rho &= \sum_{j,h=1}^3 u_{jh} |a_j\rangle \langle a_h| \sum_{n=1}^3 \rho_n^0 |a_n\rangle \langle a_n| \sum_{l,k=1}^3 u_{kl}^* |a_l\rangle \langle a_k| \\ &= \sum_{j,k=1}^3 (u \rho^0 u^\dagger)_{jk} |a_j\rangle \langle a_k| = \sum_{j,k=1}^3 \rho_{jk} |a_j\rangle \langle a_k|. \end{aligned} \quad (15)$$

where $(u^\dagger \rho u)_{mn} = \rho_n^0 \delta_{m,n}$.

Vice versa, in the dual space of the C^* -algebra any vector A has a dual partner: the base partners are:

$$|a_j\rangle \langle a_k| \mapsto \vartheta_{jk} : \vartheta_{jk}(|a_m\rangle \langle a_n|) = \delta_{j,m} \delta_{k,n}, \quad (16)$$

so that from Eq. (7) we get:

$$\vartheta_{jk}(A) = c_{jk} \quad (17)$$

and for the partner of A :

$$A \mapsto \alpha_A = \sum_{j,k=1}^3 c_{jk}^* \vartheta_{jk}, \quad (18)$$

giving:

$$\alpha_A(A) = \sum_{j,k=1}^3 c_{jk}^* c_{jk} = \text{Tr}[c^\dagger c] = \|A\|^2. \quad (19)$$

Please note that one has to rescale the above Hilbert-Schmidt norm giving $\|E\|^2 = 3$, in order to have the property $\|E\| = 1$. Given any state, positive-type functions on the unitary group \mathcal{U} are introduced as

$$\varphi(U) = \langle U \rangle_\rho = \rho(U). \quad (20)$$

They satisfy the positive semidefinite $n \times n$ matrix condition, for any n :

$$\varphi(U_j^* U_k) \geq 0, \quad \forall U_j, U_k \in \mathcal{U}, \quad j, k = 1, 2, \dots, n, \forall n. \quad (21)$$

The positive-type functions $\varphi(U)$ may be expanded in terms of tomograms $W_k(\rho, U)$ by diagonalizing the unitary matrix representing U :

$$\begin{aligned} \varphi(U) &= \rho \left(\sum_{j,h} u_{jh} |a_j\rangle \langle a_h| \right) = \sum_{j,h,k} v_{jk} \exp(i\theta_k) v_{hk}^* \rho(|a_j\rangle \langle a_h|) \\ &= \sum_k \exp(i\theta_k) \left(\sum_{j,h} v_{hk}^* \rho_{hj} v_{jk} \right) = \sum_k \exp(i\theta_k) W_k(\rho, U) \end{aligned} \quad (22)$$

Note that the tomogram component $W_k(\rho, U) = (v^\dagger \rho v)_{kk} \geq 0$ is a component of a stochastic vector: $\sum_k (v^\dagger \rho v)_{kk} = 1$.

Remark. One could have started with two vectors $|a_1\rangle, |a_2\rangle$. Then the unitary group of the resulting C^* -algebra is isomorphic to $U(2)$. In that case one can embed the

permutation group of three elements into $U(2)$, via a unitary irreducible representation, so that the corresponding positive-type functions allow for a tomographic reconstruction of the state ρ , as will be discussed in the following.

4. Group algebra

Another example of C^* -algebra is the so called group algebra, which is a tool important *per se* [39].

Following [39] we review below some properties of a group algebra, focusing first on finite groups. Given a finite group of order $K : G = G_K = \{g_1, g_2, \dots, g_K\}$, consider the complex valued functions on the group $f : G \rightarrow \mathbb{C}$. The group algebra consists of formal linear combinations of group elements:

$$A_f = \sum_{j=1}^K f(g_j) g_j, \quad (23)$$

and will be denoted by $\mathbb{C}[G]$ or \mathcal{A}_G . Each element $A \in \mathcal{A}_G$ is represented by the coefficients $f(g_i)$ of the combination and *vice versa*. We have a one-to-one correspondence between elements of the group algebra and complex valued functions on the group.

If

$$A_f = \sum_{j=1}^K f(g_j) g_j, \quad A_h = \sum_{j=1}^K h(g_j) g_j \quad (24)$$

we have $A_f + A_h = A_{f+h}$. Components of a product are obtained from

$$\begin{aligned} A_f \cdot A_h &= \sum_{j,k} f(g_j) h(g_k) g_j g_k = \sum_{j,k} f(g_j g_k^{-1}) h(g_k) g_j \\ &= \sum_{j,k} f(g_j) h(g_j^{-1} g_k) g_k = A_{f \cdot h}, \end{aligned} \quad (25)$$

where, on the algebra of group functions, the convolution product (star-product) is defined as:

$$(f \cdot h)(g_k) = \sum_{i=1}^K f(g_i) h(g_i^{-1} g_k). \quad (26)$$

The conjugate A^* of A is defined by setting $g^* = g$ and

$$A_f^* = \left(\sum_{j=1}^K f(g_j) g_j \right)^* = \sum_{j=1}^K f^*(g_j) g_j = A_{f^*}, \quad (27)$$

i.e. $f^*(g) = [f(g)]^*$.

We introduce also the transpose A^T of A by $g^T = g^{-1}$ and

$$A_f^T = \left(\sum_{j=1}^K f(g_j) g_j \right)^T = \sum_{j=1}^K f(g_j) g_j^{-1} = \sum_{j=1}^K f(g_j^{-1}) g_j = A_{f^T}, \quad (28)$$

thus $f^T(g) = f(g^{-1})$. For a product $A \cdot B$ we have $(A \cdot B)^T = B^T \cdot A^T$.

Hermitian conjugation is now defined as the composition of conjugation and transposition: $A^* = (A^T)^*$ or $g^* = g^{-1}$ and

$$A_f^* = \sum_{j=1}^K f^*(g_j) g_j^{-1} = \sum_{j=1}^K f^*(g_j^{-1}) g_j = A_{f^*}, \quad (29)$$

i. e., $f^*(g) = f(g^{-1})$. For a product $A \cdot B$ we have $(A \cdot B)^* = B^* \cdot A^*$.

All above operations are involutions, this is:

$$(A^*)^* = A, \quad (A^T)^T = A, \quad (A^*)^* = A. \quad (30)$$

We observe that only the \star -involution satisfies the condition $\Pi(A^*) = \Pi^\dagger(A)$, for any unitary representation Π of the algebra.

The trace of an element $A \in \mathcal{A}_G$ is defined by $\text{Tr}[g] = 1$ for $g = e$, the group unity, and $\text{Tr}[g] = 0$ otherwise. We have

$$\text{Tr}[A_f] = f(e). \quad (31)$$

Now we introduce the scalar product $\langle A_f, A_h \rangle$ in the group algebra \mathcal{A}_G by

$$\langle A_f, A_h \rangle = \text{Tr}[A_f^* \cdot A_h] = \sum_{j=1}^K f^*(g_j) h(g_j). \quad (32)$$

which agrees with the standard inner product on complex valued functions on G considered as vectors on \mathbb{C}^K . It follows that $\text{Tr}[A_f^* \cdot A_h] = (\text{Tr}[A_h^* \cdot A_f])^*$. It is worth to note that A^\dagger is the Hermitian conjugate of A with the scalar product we have just defined.

The associativity of the group G implies the associativity of the group algebra. The scalar product is preserved by left and right action, $g_k \mapsto L_{g_i}(g_k) = g_j g_k$, $g_j \mapsto R_{g_k}(g_i) = g_j g_k$, and similarly under conjugation $g_k \mapsto C_{g_j}(g_k) = g_j g_k g_j^{-1}$. It is also invariant under transposition $A \mapsto A^T$ and multiplication by a phase $A \mapsto \exp(i\theta) A$. Under the transformations $A \mapsto A^*$ and $A \mapsto A^*$ the scalar product goes into its complex conjugate. Moreover the transformations $A \mapsto g A g^{-1}$ and $A \mapsto A^*$ are automorphisms of the group algebra. We should also mention that a pointwise product is available:

$$A_f \circ A_h = \sum_j f(g_j) h(g_j) g_j = A_{fh} \quad (33)$$

which is called the Hadamard product.

4.1. Representations of group algebras

All irreducible representations of a finite group G of order K are finite dimensional and equivalent to unitary representations of it. If $\{D^\alpha\}$, with $\dim D^\alpha$ is an irreducible unitary representation of G , then because of Schur's Lemma, we get the orthogonality conditions:

$$\sum_{j=1}^K (D_{rs}^\alpha(g_j))^* D_{pq}^\beta(g_j) = \frac{K}{n_\alpha} \delta_{\alpha,\beta} \delta_{r,p} \delta_{s,q}, \quad (34)$$

that imply that the matrix elements $\{D_{rs}^\alpha(g_j)\}$ of the set of irreducible unitary representations of G form an orthogonal set on the algebra \mathcal{A}_G . Notice that the subspace of \mathcal{A}_G spanned by the elements (r, s) of the irreducible representation D^α is invariant under left (or right) translations, hence they define invariant subspaces of the regular representation, i.e., the canonical representation of the group G on its algebra \mathcal{A}_G by left translations. Hence all irreducible representations are contained in the regular representation, then there is a finite number of irreducible representations labelled by α and the matrix elements $\{D_{rs}^\alpha(g_j)\}$ form an orthogonal basis in the algebra of group functions:

$$f(g_j) = \sum_{\alpha} \sum_{r,s=1}^{n_{\alpha}} c_{rs}^{\alpha} D_{rs}^{\alpha}(g_j). \quad (35)$$

Moreover the dimensions n_{α} of the irreducible representations $\{D^{\alpha}\}$, satisfy the equation:

$$\sum_{\alpha} n_{\alpha}^2 = K. \quad (36)$$

One can use a unitary (reducible or irreducible) representation $U(g)$ of the group acting on an N -dimensional Hilbert space, to introduce a representation of the group algebra by means of operators on the same Hilbert space. The operator \hat{A}_f corresponding to the group algebra element A_f will be:

$$\hat{A}_f = \sum_{j=1}^K f(g_j) U(g_j). \quad (37)$$

In view of

$$U(g_j^{-1}g_l) = U(g_j^{-1})U(g_l), \quad (38)$$

one gets

$$\begin{aligned} \hat{A}_f \hat{A}_h &= \sum_{j=1}^K f(g_j) U(g_j) \sum_{l=1}^K h(g_j^{-1}g_l) U(g_j^{-1}g_l) \\ &= \sum_{j,l=1}^K f(g_j) h(g_j^{-1}g_l) U(g_l) = \sum_{l=1}^K (f \cdot h)(g_l) U(g_l) = \hat{A}_{f \cdot h}. \end{aligned} \quad (39)$$

When $U(g)$ is an irreducible representation $D^{\alpha}(g)$ the orthogonality relations (34) may be used to obtain the inversion formula

$$f(g) = \frac{n_{\alpha}}{K} \text{Tr} \left[\hat{A}_f D^{\alpha\dagger}(g) \right]. \quad (40)$$

This shows that we are in the framework of a star-product scheme, where quantizer and dequantizer operators are $D(g)$ and $D^{\dagger}(g)$ respectively [41, 42, 43].

Remark. When the group is finite, the group \mathcal{U} of unitary elements in the group algebra may be readily determined. \mathcal{U} consists, by definition, of the elements corresponding to group functions f 's satisfying the relation

$$f^{-1} = f^{\star}, \quad (41)$$

where we recall that f^* is defined as $f^*(g) = f(g^{-1})$. Condition (41) expresses unitarity at the abstract level of group algebra.

This implies that eq.(41) is equivalent to the condition of unitarity for the operator $u_f = \sum_{j=1}^K f(g_j) D(g_j)$, for any unitary irreducible representation of the group.

For finite groups the set of such representations is finite and known, so condition (41) gives explicitly \mathcal{U} . We have

$$f \in \mathcal{U} \quad \leftrightarrow \quad u_f^\alpha = \sum_{j=1}^K f(g_j) D^\alpha(g_j) \in U(n_\alpha), \forall \alpha \quad (42)$$

where D^α is an irreducible n_α -dimensional representation of the finite group, and $U(n_\alpha)$ the corresponding unitary group. When D^α varies in the set $\{D^\alpha\}$ of all irreducible representations of the finite group we get a set of $\sum_\alpha n_\alpha^2 = K$ linear inhomogeneous equations in the K variables $f(g_j)$ with known terms $u_f^1, \dots, u_f^\alpha \in U(n_1) \times \dots \times U(n_\alpha)$. The determinant of this system does not vanish because its rows are made by matrix elements $D_{mn}^\alpha(g_j)$, an orthonormal set of functions on the group. The linear system has a unique solution f_{u^1, \dots, u^α} for any given $g = u^1, \dots, u^\alpha \in U(n_1) \times \dots \times U(n_\alpha)$ and determines an isomorphism between $U(n_1) \times \dots \times U(n_\alpha)$ and \mathcal{U} :

$$f_g \cdot f_h = f_{gh}, \quad \forall g, h \in U(n_1) \times \dots \times U(n_\alpha). \quad (43)$$

For instance, in the simplest case of the group Z^2 there are only 2 representations (one dimensional), \mathcal{U} is isomorphic with $S^1 \times S^1$ and the isomorphism is given by

$$(e^{i\alpha}, e^{i\beta}) \in S^1 \times S^1 \leftrightarrow f_{\alpha, \beta} = \left(\frac{e^{i\alpha} + e^{i\beta}}{2}, \frac{e^{i\alpha} - e^{i\beta}}{2} \right). \quad (44)$$

This result can be easily obtained by solving directly eq.(41), which yields

$$\frac{1}{f^2(g_1) - f^2(g_2)} \begin{bmatrix} f(g_1) \\ -f(g_2) \end{bmatrix} = \begin{bmatrix} f^*(g_1) \\ f^*(g_2) \end{bmatrix}, \quad (45)$$

or the equivalent linear system (42) which reads:

$$\begin{bmatrix} f(g_1) + f(g_2) \\ f(g_1) - f(g_2) \end{bmatrix} = \begin{bmatrix} e^{i\alpha} \\ e^{i\beta} \end{bmatrix}. \quad (46)$$

5. Positive-type group functions

To deal with states and tomograms we need the definition of positive-type group functions, we recall the definition. Given any group G , a group function $\varphi(g)$ is of positive-type if the corresponding matrix

$$N_{jk}(\varphi) := \varphi(g_j^{-1}g_k), \quad g_j, g_k \in \{g_1, g_2, \dots, g_n\} \subseteq G \quad (47)$$

is positive semidefinite, for any n -tuple $\{g_1, g_2, \dots, g_n\}$ of elements of G , and for any $n \in \mathbb{N}$. We may call $N_{jk}(\varphi)$ the Naimark matrix of φ .

For any unitary representation of the group, $U(g)$, it is possible to define a positive-type group function by means of a pure state corresponding to the vector ξ :

$$\varphi_\xi^U(g) := (\xi, U(g)\xi) = \text{Tr}[\rho_\xi U(g)] . \quad (48)$$

Here, ρ_ξ is the density state corresponding to ξ . In fact, the quadratic form

$$\begin{aligned} \sum_{j,k=1}^n \lambda_j^* \lambda_k \text{Tr}[\rho_\xi U(g_j^{-1} g_k)] &= \text{Tr} \left[\rho_\xi \sum_{j=1}^n \lambda_j^* U^\dagger(g_j) \sum_{k=1}^n \lambda_k U(g_k) \right] \\ &= \text{Tr}[\rho_\xi V^\dagger V] \geq 0, \end{aligned} \quad (49)$$

where the λ 's are arbitrary complex numbers, is positive semidefinite.

The above form can be generalized by using any density state ρ instead of a pure one ρ_ξ , and this will be very useful in the tomographic framework.

It should be stressed here that the form of Eq.(48) is canonical. Because of Naimark's representation theorem [39], for any positive-type group function $\varphi(g)$ there exist a Hilbert space, a unitary representation of the group $U(g)$ and a cyclic vector ξ such that

$$\varphi(g) = (\xi, U(g)\xi) . \quad (50)$$

We recall that a vector ξ is called cyclic if the set $\{U(g)\xi \mid g \in G\}$ spans the Hilbert space.

Notably, for a finite group, the positivity of a group function may be checked considering only one Naimark matrix, constructed with all the elements of the group. We have the following

Proposition 1 *A group function ψ defined on a finite group $G_K = \{g_1, g_2, \dots, g_K\}$ of order K is of positive type iff the $K \times K$ -matrix:*

$$N(\psi)_{ij} = \psi(g_i^{-1} g_j), \quad i, j = 1, \dots, K \quad (51)$$

is positive semidefinite.

Proof: In fact, consider the Naimark matrix $N(\psi)_{ij}$ of order $K+1$ obtained by adding a repeated element $g_{K+1} = g_h$ to $\{g_1, g_2, \dots, g_K\}$. Then, $N(\psi)_{ij}$ has two equal rows. So, $\det N(\psi)_{ij} = 0$. By induction, $\det N(\psi)_{ij} = 0$ for $g_i, g_j \in \{g_1, \dots, g_K, \dots, g_{K+p}\}$, $\forall p$, and the proposition is proven.

6. Finite groups and tomography

We introduce a tomography for any density state ρ by means of a positive type function on G , defined as:

$$\varphi_\rho^D(g) := \text{Tr}[\rho D(g)] \quad (52)$$

where $\{D(g)\}$ is a unitary representation of G on the Hilbert space where the density state ρ is defined.

In this section, we consider again a finite group of order $K : G = G_K = \{g_1, g_2, \dots, g_K\}$. Then we may suppose the representation D to be n -dimensional ($n^2 < K$ if D is irreducible.) For any group element g the corresponding representative matrix $D(g)$ can be put in the form of a diagonal unitary matrix d_g by means of a unitary matrix V_g :

$$D(g) = V_g d_g V_g^\dagger, \quad d_g = \text{diag} [e^{i\theta_1(g)}, \dots, e^{i\theta_n(g)}]. \quad (53)$$

We observe that, in general, neither d_g nor V_g separately are group representations. Moreover, V_g is not uniquely determined for, if C_r, C_l are unitary matrices commuting with d_g and ρ respectively, we have:

$$\begin{aligned} \varphi_\rho^D(g) &:= \text{Tr} [\rho V_g d_g V_g^\dagger] \\ &= \text{Tr} [C_l^\dagger \rho C_l V_g d_g C_r^\dagger V_g^\dagger] \\ &= \text{Tr} [(C_l V_g C_r)^\dagger \rho (C_l V_g C_r) d_g], \end{aligned} \quad (54)$$

so that this ambiguity does not affect the associate function, and we may write unambiguously:

$$\begin{aligned} \varphi_\rho^D(g) &= \text{Tr} [d_g (V_g^\dagger \rho V_g)] \\ &= \sum_{m=1}^n e^{i\theta_m(g)} (V_g^\dagger \rho V_g)_{mm} \\ &=: \sum_{m=1}^n e^{i\theta_m(g)} W_m(g, \rho). \end{aligned} \quad (55)$$

In the last equation, we have introduced the components

$$W_m(g, \rho) := (V_g^\dagger \rho V_g)_{mm} \quad (m = 1, \dots, n) \quad (56)$$

of the vector $\mathbf{W}(g, \rho)$ defining the tomogram of ρ in the chosen representation of G_K . We note that, as $V_g^\dagger \rho V_g$ is again a density state, the tomogram is by definition a stochastic vector, i.e.:

$$\sum_{m=1}^n W_m(g, \rho) = \sum_{m=1}^n (V_g^\dagger \rho V_g)_{mm} = \text{Tr}[\rho] = 1, \quad (57)$$

$$W_m(g, \rho) \geq 0 \quad (m = 1, \dots, n), \forall g \in G_K. \quad (58)$$

The knowledge of the tomograms $\{\mathbf{W}(g_j, \rho)\}_{j=1}^K$ allows for reconstructing the density state. In fact, as the diagonal matrices d_g 's depend only on the representation D and are supposed to be known, the function φ_ρ^D is readily obtained as:

$$\varphi_\rho^D(g_j) = \sum_{m=1}^n e^{i\theta_m(g_j)} W_m(g_j, \rho). \quad (59)$$

Then the state is given by the reconstruction formula:

$$\frac{n}{K} \sum_{j=1}^K (\varphi_\rho^D(g_j))^* D(g_j) = \rho, \quad (60)$$

which is based on the orthogonality relations of the matrix elements of $D(g)$:

$$\begin{aligned} \frac{n}{K} \sum_{j=1}^K (\varphi_\rho^D(g_j))^* D_{rs}(g_j) &= \frac{n}{K} \sum_{j=1}^K \text{Tr} [\rho^* D^*(g_j)] D_{rs}(g_j) \\ &= \sum_{q,m=1}^n \rho_{qm}^* \frac{n}{K} \sum_{j=1}^K D_{mq}^*(g_j) D_{rs}(g_j) = \sum_{q,m=1}^n \rho_{qm}^* \delta_{m,r} \delta_{q,s} = \rho_{sr}^* = \rho_{rs}. \end{aligned} \quad (61)$$

Now, suppose that φ is any positive type function on G_K . We recall that, by Naimark's theorem, there exist a Hilbert space acted upon by a unitary representation U of G_K and a cyclic vector ξ such that:

$$\varphi(g_j) = (\xi, U(g_j)\xi) = \text{Tr} [\rho_\xi U(g_j)]. \quad (62)$$

In general the above representation U results reducible as a direct sum of all the irreducible representations $\{D^\alpha\}$, $\dim D^\alpha$, $(\sum_\alpha n_\alpha^2 = K)$, of the group, each block D^α with multiplicity m_α :

$$U = \bigoplus_{\alpha} \bigoplus_{s=1}^{m_\alpha} D_s^\alpha. \quad (63)$$

Out of the matrix representing ρ_ξ one can extract the same blocks of the reduction of U , to construct a new matrix $\tilde{\rho}$, with the remaining entries zero. Moreover, $\tilde{\rho}$ is still a state, as the determinants of its blocks are principal minors of ρ_ξ . They are nonzero because ρ_ξ is cyclic. Then, by construction, the function $\text{Tr} [\tilde{\rho} U(g_j)]$ coincides with the above function $\varphi(g_j)$, i.e.:

$$\varphi(g_j) = \text{Tr} [\rho_\xi U(g_j)] = \text{Tr} [\tilde{\rho} U(g_j)]. \quad (64)$$

Now we sum together the blocks ρ_s^α of $\tilde{\rho}$ associated to the same D^α :

$$\tilde{\rho}^\alpha = \sum_{s=1}^{m_\alpha} \rho_s^\alpha \quad (65)$$

and finally we can write

$$\varphi(g_j) = \text{Tr} [\tilde{\rho} U(g_j)] = \sum_{\alpha} \text{Tr} [\tilde{\rho}^\alpha D^\alpha(g_j)]. \quad (66)$$

The function φ is normalized, i.e., on the identity element e of the group, $\varphi(e) = 1$. Then $\tilde{\rho}^\alpha$ can be written as $\tilde{\rho}^\alpha = \gamma^\alpha \rho^\alpha$, where $0 \leq \gamma^\alpha \leq 1$, $\sum_\alpha \gamma^\alpha = 1$ and ρ^α is a density state.

So, we have proven:

Proposition 2 *Any positive-type function φ on G_K can be decomposed as a convex sum of the positive-type functions φ^α 's related tomographically to the irreducible representations D^α 's of the group:*

$$\varphi(g_j) = \sum_{\alpha} \gamma^\alpha \varphi^\alpha(g_j), \quad \varphi^\alpha(g_j) = \text{Tr} [\rho^\alpha D^\alpha(g_j)]. \quad (67)$$

We remark that any φ^α can be written, again using the Naimark theorem, in terms of a pure cyclic state and a representation U^α as

$$\varphi^\alpha(g_j) = (\xi^\alpha, U^\alpha(g_j)\xi^\alpha) = \text{Tr} [\rho_{\xi^\alpha} U^\alpha(g_j)]. \quad (68)$$

So, the question arises to relate U^α to D^α and ρ_{ξ^α} to ρ^α . Dropping the label α , we can state the following:

Proposition 3 *If the density state ρ is of rank r , the above representation U results reducible as a direct sum of r blocks, each one unitarily equivalent to the irreducible representation D . Then, after a possible rearrangement, $U = \bigoplus_{s=1}^r D_s$. The state ρ can be used to obtain a pure state ρ_ξ , cyclic for U .*

The proof amounts to the GNS construction. By using the harmonic expansion of the group functions in the basis of the matrix elements $D_{qp}^\beta(g_j)$ of all the irreducible representations $\{D^\beta\}$, $\dim D_\beta^\beta$, of the group, and bearing in mind that the dimension of the given D is n , we may write:

$$\varphi(g_j) = \sqrt{\frac{n}{K}} \sum_{q,p=1}^n \varphi_{qp} D_{qp}(g_j), \quad (69)$$

where

$$\varphi_{qp} = \sqrt{\frac{n_\alpha}{K}} \sum_{j=1}^K \varphi(g_j) D_{qp}^*(g_j). \quad (70)$$

Then, the convolution product on the algebra of group functions (26) for X, Y reads

$$(X \cdot Y)(g_j) = \sum_{i=1}^K X(g_i) Y(g_i^{-1} g_j) \quad (71)$$

and may be expanded as

$$\begin{aligned} (X \cdot Y)(g_j) &= \sum_{i=1}^K \sum_{\alpha} \sqrt{\frac{n_\alpha}{K}} \sum_{q,p=1}^{n_\alpha} X_{qp}^\alpha D_{qp}^\alpha(g_i) \times \\ &\quad \times \sum_{\beta} \sqrt{\frac{n_\beta}{K}} \sum_{m,s=1}^{n_\beta} Y_{ms}^\beta D_{ms}^\beta(g_i^{-1} g_j) \end{aligned} \quad (72)$$

From

$$D_{ms}^\beta(g_i^{-1} g_j) = \sum_{r=1}^{n_\beta} D_{mr}^\beta(g_i^{-1}) D_{rs}^\beta(g_j) = \sum_{r=1}^{n_\beta} (D_{rm}^\beta(g_i))^* D_{rs}^\beta(g_j) \quad (73)$$

and the orthogonality relations Eq. (34), the convolution product may be written as

$$(X \cdot Y)(g_j) = \sum_{\alpha} \sum_{q,p,s=1}^{n_\alpha} X_{qp}^\alpha Y_{ps}^\alpha D_{qs}^\alpha(g_j) = \sum_{\alpha} \sum_{q,s=1}^{n_\alpha} (XY)_{qs}^\alpha D_{qs}^\alpha(g_j). \quad (74)$$

By introducing the function

$$X^\dagger(g) := X^*(g^{-1}) = \sum_{\alpha} \sqrt{\frac{n_\alpha}{K}} \sum_{q,p=1}^{n_\alpha} (X_{pq}^\alpha)^* D_{qp}^\alpha(g), \quad (75)$$

we define a seminorm

$$\begin{aligned}
 F(X^\dagger \cdot X) &= \sum_{j=1}^K (X^* \cdot X)(g_j)(\varphi(g_j))^* \\
 &= \sum_{j=1}^K \sum_{\alpha} \sum_{q,p,s=1}^{n_\alpha} (X_{pq}^\alpha)^* X_{ps}^\alpha D_{qs}^\alpha(g_j) \sum_{m,r=1}^{n_{\alpha_0}} \varphi_{mr}^*(D_{mr}(g_j))^* \\
 &= \frac{K}{n} \sum_{q,p,s=1}^n X_{pq}^* X_{ps} \varphi_{qs}^*.
 \end{aligned} \tag{76}$$

Without any loss of generality, we may suppose the density state is diagonal: $\rho = \text{diag}(\rho_1, \rho_2, \dots, \rho_n)$. For, upon diagonalization,

$$\varphi(g) := \text{Tr}[\rho D(g)] = \text{Tr}[\text{diag}(\rho_1, \rho_2, \dots, \rho_n) V^\dagger D(g) V] \tag{77}$$

and we could choose in the previous discussion $V^\dagger D(g) V$ instead of $D(g)$ from the very beginning. Then

$$\varphi_{qs}^* = \rho_q \delta_{q,s} \tag{78}$$

and the seminorm reads:

$$F(X^\dagger \cdot X) = \frac{K}{n} \sum_{q=1}^n \left(\sum_{p=1}^n |X_{pq}|^2 \right) \rho_q = \frac{K}{n} \sum_{q=1}^n \|\mathbf{X}_q\|^2 \rho_q. \tag{79}$$

where the vector \mathbf{X}_q is the q -th column of the matrix of coefficients of $D(g)$ in the harmonic expansion of $(X^\dagger \cdot X)$.

Now, suppose the density state ρ has rank r , with non-zero entries

$$\{\rho_{s_1}, \rho_{s_2}, \dots, \rho_{s_r}\}. \tag{80}$$

Then, in view of eq.(79), the seminorm kernel $\mathcal{F}_0 = \{X : F(X^\dagger \cdot X) = 0\}$ is given by the functions X such that the columns

$$\{\mathbf{X}_{s_1}, \mathbf{X}_{s_2}, \dots, \mathbf{X}_{s_r}\}$$

of the representative matrix (X_{pq}) vanish. So, F is a norm on the quotient $\mathcal{F}/\mathcal{F}_0$ of the algebra of group functions with respect to the kernel. Equivalence classes are labelled by the entries of the columns $\{\mathbf{X}_{s_1}, \mathbf{X}_{s_2}, \dots, \mathbf{X}_{s_r}\}$. A class representative can be chosen with vanishing expansion coefficients but those of the above columns of the matrix (X_{pq}) , we denote it as $X_{\{\mathbf{X}_{s_1}, \mathbf{X}_{s_2}, \dots, \mathbf{X}_{s_r}\}}$. In other words, we have:

$$X_{\{\mathbf{X}_{s_1}, \mathbf{X}_{s_2}, \dots, \mathbf{X}_{s_r}\}}(g) = \sqrt{\frac{n}{K}} \sum_{p=1}^n \sum_{q=1}^r X_{ps_q} D_{ps_q}(g)$$

The r columns labeling the classes determine a Hilbert space of dimension rn , and a corresponding group representation U^* may be defined as

$$\begin{aligned}
 (U^*(h) X_{\{\mathbf{X}_{s_1}, \mathbf{X}_{s_2}, \dots, \mathbf{X}_{s_r}\}})(g) &:= X_{\{\mathbf{X}_{s_1}, \mathbf{X}_{s_2}, \dots, \mathbf{X}_{s_r}\}}(h^{-1}g) \\
 &= \sqrt{\frac{n}{K}} \sum_{m=1}^n \sum_{q=1}^r \left(\sum_{p=1}^n D_{mp}^*(h) X_{ps_q} \right) D_{ms_q}(g) \\
 &= X_{\{D^*(h)\mathbf{X}_{s_1}, D^*(h)\mathbf{X}_{s_2}, \dots, D^*(h)\mathbf{X}_{s_r}\}}(g).
 \end{aligned} \tag{81}$$

In other words, we have $U^* = \oplus_{s=1}^r D_s^*$, or $U = \oplus_{s=1}^r D_s$, and the sum has r terms.

We can use generalized orthogonality relations, to get

$$\frac{n}{K} \sum_{j=1}^K (\varphi(g_j))^* U(g_j) = \bigoplus_{s=1}^r \rho_s, \quad \rho_s = \rho \quad \forall s \quad (82)$$

where the sum, which has r terms equal to ρ , is not a density state any further.

Now, we construct a rn -dimensional column vector state $\xi = \{\xi_m\}_m$ by using the nonzero rows of ρ :

$$\xi_m = \sum_{j=1}^r \sqrt{\rho_{s_j}} \delta_{m, n(j-1)+s_j}, \quad m = 1, 2, \dots, rn \quad (83)$$

which defines a pure state ρ_ξ , cyclic for U and such that $(\xi, U(g)\xi) = \text{Tr}[\rho D(g)]$.

This completes the proof.

7. The example of S_3 : the permutation group of three elements

We examine now the group S_3 of permutation of three elements, which is isomorphic to the group of symmetries of a triangle, to show how the considerations of the previous sections appear in a concrete example.

S_3 has six elements $\{g_k : k = 1, \dots, 6\}$ with a law of multiplication encoded in the following table

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \quad (84)$$

from which one can obtain the group law via

$$g_i g_k = g_{R_{i,k}}. \quad (85)$$

For example, the table gives $2 \cdot 3 = 1$. The inverse elements are given by

$$g_1^{-1} = g_1, g_2^{-1} = g_3, g_3^{-1} = g_2, g_4^{-1} = g_4, g_5^{-1} = g_5, g_6^{-1} = g_6. \quad (86)$$

For example, from this rule we get $2^{-1} = 3$. From tables 84 and 86 one get the table for $g_i^{-1} g_k$ which reads:

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}. \quad (87)$$

The space of group functions f on S_3 is isomorphic to \mathbb{C}^6 : $f_k := f(g_k) \in \mathbb{C}$; $k = 1, \dots, 6$. The Naimark matrix of f is the 6×6 -matrix with entries $f(g_k^{-1}g_m) = f(g_{L_{km}})$ obtained by computing f in the points labelled by L .

The left regular representation D^L of the group acting on functions f is defined as

$$(D^L(g_k)f)(g_m) = f_{g_k}(g_m) = f(g_k^{-1}g_m) = f(g_{L_{km}}) \quad (88)$$

therefore in the k -th row of the matrix Lf one finds the six values of $D^L(g_k)f$; the left regular representation is made by the following six 6×6 -matrices $D^L(g_k)_{mn} = \delta_{m, L_{kn}}$. In analogous way, by using the transpose of R instead of L , one gets the right action. The characters of the left regular representation are easily computed to be $\chi^L = (6, 0, 0, 0, 0, 0)$.

S_3 has three unitary irreducible representations, $D^0 : \{1, 1, 1, 1, 1, 1\}$, with character $\chi^0 : \{1, 1, 1, 1, 1, 1\}$, $D^1 = \{1, 1, 1, -1, -1, -1\}$, with character $\chi^1 = \{1, 1, 1, -1, -1, -1\}$, and $D^2 :$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \begin{pmatrix} \lambda^2 & 0 \\ 0 & \bar{\lambda}^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda^2 \\ \bar{\lambda}^2 & 0 \end{pmatrix} \right\}$$

where

$$\lambda = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

with character

$$\chi^2 = \{2, 2\operatorname{Re}\lambda, 2\operatorname{Re}\lambda^2, 0, 0, 0\} = \{2, -1, -1, 0, 0, 0\}.$$

The character $\chi^L = (6, 0, 0, 0, 0, 0)$ can be decomposed as

$$\chi^L = \chi^0 + \chi^1 + 2\chi^2$$

and therefore the left regular representation is unitarily equivalent to

$$D^0 \oplus D^1 \oplus D^2 \oplus D^2.$$

Examples of positive-type functions are the diagonal elements of any unitary representation and any linear combination of them with positive coefficient. Characters are therefore positive-type functions. For instance the Naimark matrix of χ^2 is

$$N(\chi^2) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \quad (89)$$

and has eigenvalues: $0, 0, 3, 3, 3, 3$.

One can write the most general positive-type function on the group. The most general $N(f)$, which takes care only of Hermiticity conditions, must be proportional to the matrix

$$\begin{bmatrix} 1 & a + i b & a - i b & r & s & t \\ a - i b & 1 & a + i b & t & r & s \\ a + i b & a - i b & 1 & s & t & r \\ r & t & s & 1 & a - i b & a + i b \\ s & r & t & a + i b & 1 & a - i b \\ t & s & r & a - i b & a + i b & 1 \end{bmatrix} \quad (90)$$

where a, b, r, s, t are real. The different eigenvalues are

$$2a + 1 \pm (r + s + t), 1 - a \pm \sqrt{3b^2 - rt - st - rs + r^2 + s^2 + t^2}. \quad (91)$$

The function

$$f = (1, a + ib, a - ib, r, s, t) \quad (92)$$

is of positive-type *iff* these eigenvalues are nonnegative.

Let us consider the matrix M constructed by taking as rows the matrix elements with the same row label in all the irreducible group representation matrices, normalized to be a unity norm vector. The matrix M is unitary, due to the the orthogonality relations (34). It reads

$$M = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{\lambda}{\sqrt{3}} & \frac{\lambda^2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{\lambda}{\sqrt{3}} & \frac{\lambda^2}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{\bar{\lambda}}{\sqrt{3}} & \frac{\bar{\lambda}^2}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\bar{\lambda}}{\sqrt{3}} & \frac{\bar{\lambda}^2}{\sqrt{3}} & 0 & 0 & 0 \end{bmatrix}.$$

The matrix M diagonalizes the Naimark matrix of χ^2 :

$$M^\dagger N(\chi^2) M = \text{diag}[0, 0, 3, 3, 3, 3]. \quad (93)$$

Using the orthogonality relations, it can be shown that, for any finite or compact group, the Naimark matrix of characters is diagonalized by the corresponding M matrix.

Finally, recalling the remark in the end of subsection 4.1, we note that the above explicit form of the unitary matrix M solves the problem of determining the unitary elements f 's in the group algebra of S_3 as:

$$f = M^\dagger (u^0, u^1, u_{11}^2, u_{12}^2, u_{21}^2, u_{22}^2)^\top \quad (94)$$

where $u^0, u^1 \in U(1)$ and the matrix u^2 belongs to $U(2)$.

7.1. The group algebra of a compact Lie group

The notion of group algebra can be extended to compact Lie groups. The essential aspect for the definition of a group algebra is the existence of an (bi-)invariant measure, the Haar measure dg . Thus any continuous function $f : G \rightarrow \mathbb{C}$ on a compact Lie group is integrable with respect to the Haar measure:

$$\int_G f(g) dg < \infty. \quad (95)$$

and the integral is invariant under left as well as right actions:

$$\int_G f(gh) dg = \int_G f(gh) dh = \int_G f(g) dg. \quad (96)$$

The measure dg is normalized in such a way that the volume of the group is one. We will consider the algebra \mathcal{A}_G consisting on all integrable functions on the group G , i.e., $\mathcal{A}_G = L^1(G, dg)$, together with the convolution product. Thus if A is the element on \mathcal{A}_G represented by the function f_A , we will have that the element $A \cdot B$ is represented by the function

$$\int_G f_A(h) f_B(h^{-1}g) dh = \int_G f_A(gh^{-1}) f_B(h) dh = f_{A \cdot B}(g), \quad (97)$$

along with

$$\text{Tr} [A^\dagger B] = \int_G f_A^*(g) f_B(g) dg. \quad (98)$$

Other properties of the finite group algebra are extended very easily in terms of representing functions.

For instance, consider the group $U(1)$, with group manifold the circle $0 \leq \theta < 2\pi$. The Abelian group $U(1)$ has irreducible one-dimensional representations labelled by integers:

$$D^m : \theta \mapsto \exp(im\theta), \quad m \in \mathbb{Z}, \quad (99)$$

and their characters are: $\chi^m(\theta) = \exp(im\theta)$.

The corresponding M matrix has discrete row and continuous column labelling indices

$$(M_{m\theta}) = \frac{1}{\sqrt{2\pi}} (\exp(im\theta)). \quad (100)$$

Of course, it is unitary, that is

$$\sum_\theta (M_{m\theta}) (M_{m'\theta}^*) = \frac{1}{2\pi} \int_0^{2\pi} \exp(i(m - m')\theta) d\theta = \delta_{m,m'} \quad (101)$$

and

$$\sum_m (M_{m\theta}^*) (M_{m\theta'}) = \frac{1}{2\pi} \sum_m \exp(im(\theta - \theta')) = \delta(\theta - \theta'). \quad (102)$$

The Naimark matrix of χ^m has elements $\exp[i m(\theta' - \theta)]/2\pi$ and is diagonalized by M :

$$\begin{aligned} & (M^\dagger N(\chi^m) M)_{m_1 m_2} \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \exp[i(m_1 - m)\theta + i(m - m_2)\theta'] d\theta d\theta' = \delta_{m_1, m_2}. \end{aligned} \quad (103)$$

7.2. States and tomograms in two dimensions

States in two dimensions are parametrized by points of the 3-dimensional solid sphere

$$\rho = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x\sigma_x + y\sigma_y + z\sigma_z = \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} \quad (104)$$

where

$$\sigma_x = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (105)$$

are the Pauli matrices. The eigenvalues of ρ are

$$\rho_{\mp} = \frac{1}{2} (1 \mp r) \quad (106)$$

where by the positivity condition $r^2 = x^2 + y^2 + z^2 \leq 1$, so (x, y, z) is a point of a ball (Bloch sphere) of radius 1 centered at the origin and the pure states are points on the surface $x^2 + y^2 + z^2 = 1$.

The diagonal matrices d_g 's for D^2 are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}, \quad \begin{bmatrix} \lambda^2 & 0 \\ 0 & \bar{\lambda}^2 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}_{j=4,5,6}, \quad (107)$$

while the diagonalizing V_g 's, such that $V_g^\dagger D^2(g) V_g = d_g$, are respectively

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{j=1,2,3}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \\ 1 & 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{i\frac{4\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & 1 \end{bmatrix}. \quad (108)$$

The V_g 's are determined up to phases, one for each column; tomograms $(V_g^\dagger \rho V_g)_{mm}$ are invariant under the change of these phases. The first $V_g = V_e$ is an arbitrary unitary matrix, here chosen as the identity. At the point $g = e$ the tomogram is an arbitrary stochastic vector: this is in agreement with the probabilistic interpretation of the tomogram as probability of getting the eigenvalues of $D(e)$ in a measure.

The matrices $\left\{ V_{g_j}^\dagger \rho V_{g_j} \right\}_{j=1, \dots, 6}$ are

$$\begin{aligned} V_{g_j}^\dagger \rho V_{g_j} &= \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}, \quad j = 1, 2, 3 \\ V_{g_j}^\dagger \rho V_{g_j} &= \frac{1}{2} \begin{bmatrix} 1-x & -(z-iy) \\ -(z+iy) & 1+x \end{bmatrix}, \quad j = 4 \end{aligned} \quad (109)$$

$$V_{g_j}^\dagger \rho V_{g_j} = \frac{1}{2} \begin{bmatrix} 1 + \frac{1}{2}(x + \sqrt{3}y) & -z - \frac{1}{2}i(y - \sqrt{3}x) \\ -z + \frac{1}{2}i(y - \sqrt{3}x) & 1 - \frac{1}{2}(x + \sqrt{3}y) \end{bmatrix}, \quad j = 5,$$

$$V_{g_j}^\dagger \rho V_{g_j} = \frac{1}{2} \begin{bmatrix} 1 + \frac{1}{2}(x - \sqrt{3}y) & -z - \frac{1}{2}i(y + \sqrt{3}x) \\ -z + \frac{1}{2}i(y + \sqrt{3}x) & 1 - \frac{1}{2}(x - \sqrt{3}y) \end{bmatrix}, \quad j = 6.$$

The tomograms $(V_{g_j}^\dagger \rho V_{g_j})_{mm}$ for a generic two dimensional state with respect to the representation D^2 are the stochastic vectors

$$\begin{aligned} & \frac{1}{2} \begin{bmatrix} 1+z \\ 1-z \end{bmatrix}_{j=1,2,3}, \frac{1}{2} \begin{bmatrix} 1-x \\ 1+x \end{bmatrix}_{j=4}, \\ & \frac{1}{2} \begin{bmatrix} 1 + \frac{1}{2}(x + \sqrt{3}y) \\ 1 - \frac{1}{2}(x + \sqrt{3}y) \end{bmatrix}_{j=5}, \frac{1}{2} \begin{bmatrix} 1 + \frac{1}{2}(x - \sqrt{3}y) \\ 1 - \frac{1}{2}(x - \sqrt{3}y) \end{bmatrix}_{j=6} \end{aligned} \quad (110)$$

7.3. Positive-type functions

In view of the Proposition 2, any positive group function has the form $\varphi_\rho = \text{Tr}[\rho D]$, with $D = D^0 \oplus D^1 \oplus D^2$ and ρ decomposes accordingly. The 4×4 -density state ρ has the form

$$\rho = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \frac{1}{2}\gamma(1+z) & \frac{1}{2}\gamma(x-iy) \\ 0 & 0 & \frac{1}{2}\gamma(x+iy) & \frac{1}{2}\gamma(\frac{1}{2}-z) \end{bmatrix}, \quad (111)$$

where $\alpha, \beta, \gamma \geq 0$, with $\alpha + \beta + \gamma = 1$, while the positive type function $\varphi_\rho = \text{Tr}[\rho D]$ has values

$$\varphi_\rho = \begin{bmatrix} 1 \\ \alpha + \beta - \frac{1}{2}\gamma(1-i\sqrt{3}z) \\ \alpha + \beta - \frac{1}{2}\gamma(1+i\sqrt{3}z) \\ \alpha - \beta + \gamma x \\ \alpha - \beta - \frac{1}{2}\gamma(x + \sqrt{3}y) \\ \alpha - \beta - \frac{1}{2}\gamma(x - \sqrt{3}y) \end{bmatrix}. \quad (112)$$

This vector gives explicitly the form previously obtained in eq. (92). For $\alpha, \beta = 0$ this gives positive functions when only D^2 is present.

8. Tomogram associated with $U(n)$ groups

In this section we introduce the tomograms of states associating the tomographic probabilities with the group $U(n)$ and other compact Lie groups G . Since $U(n)$ can be factorized as $U(1) \times SU(n)$ up to a quotient by \mathbb{Z}_n , we will be mainly concerned with $SU(n)$.

As a general remark, we observe that all the previous results can be straightforwardly extended to the present case. The diagonalization procedure leading to the tomographic scheme for finite groups is recovered by means of the theory of

maximal tori for compact groups. In fact, the diagonalization procedure provides a set of eigen-projectors, containing a family of rank-one projectors which is tomographic, i.e., it is a resolution of the identity. In the compact group case, the tomographic family is obtained by group action on a fiducial set of rank-one projectors, obtained by the eigenvectors of a complete set of commuting observables.

We begin with a review of some results on compact Lie groups G that will be needed in what follows [47].

Any element g of G lies on a one-parameter subgroup L which needs not be compact, and whose closure is a torus T . Every such torus T is contained in a maximal torus T_{\max} , so that any element of the group belongs to a maximal toroid at least. All maximal tori are conjugated: if T_{\max} and T'_{\max} are maximal tori, there exists an element g such that $T'_{\max} = gT_{\max}g^{-1}$. So, maximal tori have the same dimension r , the rank of the group G . Besides, G may be obtained by conjugating a fixed maximal torus T_{\max} by all elements of G , or, more simply, by representative elements g of cosets $[g]$ of G/T_{\max} :

$$G = \bigcup_{g \in G} gT_{\max}g^{-1} = \bigcup_{[g] \in G/T_{\max}} gT_{\max}g^{-1} \quad (113)$$

An element t of T_{\max} is called regular if it does not belong to any other maximal torus, otherwise the element is singular. In other words, t is singular if and only if there exists $g \notin T_{\max}$ such that $gtg^{-1} = t$, in particular the unity of G is singular.

From a tomographic point of view, it is necessary to describe the previous results by using the Lie algebra \mathfrak{g} of G and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, which is mapped in a maximal torus $T_{\mathfrak{h}}$ by the exponential map. To characterize the singular elements t of $T_{\mathfrak{h}}$ we introduce a basis of d generators of \mathfrak{g} : $E_1, \dots, E_{d-r}, H_1, \dots, H_r$. Upon putting $t = \exp \xi^b H_b$, $g = \exp \eta^a E_a$, (hereafter we adopt Einstein summation convention) we look for solutions of $gtg^{-1} = t$, $g \notin T_{\mathfrak{h}}$, at the level of Lie algebra, in the form

$$[\eta^a E_a, \xi^b H_b] = 0. \quad (114)$$

This amounts to

$$C_{a,b}^{a'} \xi^b \eta^a = 0; a, a' = 1, \dots, d-r, \quad (115)$$

where $C_{a,b}^{a'}$ are structure constants of the algebra of the group G . The above square linear system in the unknowns $\{\eta^a\}$ yields the commutant, external to the Cartan subalgebra, of the given element $\xi^b H_b$. Non-trivial solutions correspond to singular elements $t = \exp(\xi^b H_b)$. If the compact Lie group G is semisimple we can identify its Lie algebra and its dual by means of the Killing–Cartan form. The dimension of the orbit of the (co-)adjoint action of the group on its Lie algebra through a singular point $\xi^b H_b$ is smaller than that of the orbit through a regular point, which is $d-r$. The same holds for the action of the group on itself by conjugation. We recall that all the co-adjoint orbits, both regular and singular, are symplectic manifolds, hence endowed with invariant measures. Besides, from a measure theoretical point of view, the set of all singular orbits has zero Haar measure in the group. As a consequence, integration of functions on the group may be performed via Fubini's theorem, integrating over a

maximal torus $T_{\mathfrak{h}}$ and the integral over a regular orbit through t , times a Jacobian taking into account the dependence on t . Quite generally, this Jacobian can be evaluated for any compact Lie group [48].

Quantum tomography requires the use of an irreducible unitary group representation $D(g)$. Assume D is the defining representation of $G = SU(n)$. Then the Cartan subalgebra generators $\{H_b\}$ become a complete set of commuting observables of a physical system. From the previous analysis, we know that the spectrum degeneracy of $\xi^b H_b$ is even for singular points.

The adjoint action on the maximal torus gives rise to the family of unitary operators $D(g) \exp(i\xi^b H_b) D^\dagger(g)$. By decomposing the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e}$ as a direct sum of orthogonal subspaces, and choosing accordingly the basis $H_1, \dots, H_r, E_1, \dots, E_{d-r}$, we observe that the elements $\exp \eta^a E_a$ parametrize $G/T_{\mathfrak{h}}$ so that $D(g) \exp(i\xi^b H_b) D^\dagger(g)$ can be parametrized by (ξ^b, η^a) , i.e.,

$$D(g) \exp(i\xi^b H_b) D^\dagger(g) = D(\tilde{g}), \quad (116)$$

where $\tilde{g} = (\xi^b, \eta^a)$ covers almost everywhere the whole group G . In other words, $D(\tilde{g})$ is diagonalized by $D(g)$ and is iso-spectral with $\exp(i\xi^b H_b)$. Both these matrices belong to the representation, in contrast with the finite group case, where the diagonalizing matrix V_{g_k} and the diagonal matrix d_{g_k} do not belong to the representation.

We note that, as $D(g)$ diagonalizes $D(\tilde{g})$ for any $g \in [g]$, one can choose $g = (0, \eta^a)$ to avoid redundancies. The above invariant integration on the group may be performed according to that parametrization.

By using the projector valued measure (PVM) $\Pi(\xi^b)(\cdot)$ associated to the Hermitian operator $\xi^b H_b$, the spectral decomposition of $D(\tilde{g})$ may be written as

$$D(\tilde{g}) = \int_{\mathbb{R}} e^{i\lambda} \exp(i\eta^a E_a) \Pi(\xi^b)(d\lambda) \exp(-i\eta^a E_a). \quad (117)$$

By means of a density state of a physical system ρ we define a positive-type group function $\varphi(\tilde{g}) = \text{Tr}[\rho D(\tilde{g})]$ in terms of a probability measure \mathcal{M}_ρ :

$$\begin{aligned} \varphi(\tilde{g}) &= \int_{\mathbb{R}} e^{i\lambda} \text{Tr}[\rho \exp(i\eta^a E_a) \Pi(\xi^b)(d\lambda) \exp(-i\eta^a E_a)] \\ &= \int_{\mathbb{R}} e^{i\lambda} \mathcal{M}_\rho(\xi^b, \eta^a)(d\lambda). \end{aligned} \quad (118)$$

The probability measure \mathcal{M}_ρ , which is labelled by $\tilde{g} = (\xi^b, \eta^a)$, is related to the tomogram associated to the density state ρ , in the tomographic scheme based on the group G .

More precisely,

$$\int_B \mathcal{M}_\rho(\xi^b, \eta^a)(d\lambda) \quad (119)$$

is the probability that a measure of the observable $\xi^b H_b$ in the rotated state $\exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a)$ belongs to the Borel set B of the real line. As a consequence:

$$\int_B \mathcal{M}_\rho(\xi^b, \eta^a)(d\lambda) = \int_{kB} \mathcal{M}_\rho(k\xi^b, \eta^a)(d\lambda') = \int_B |k| \mathcal{M}_\rho(k\xi^b, \eta^a)(d\lambda) \quad (120)$$

and we get the homogeneity property

$$\mathcal{M}_\rho(k\xi^b, \eta^a; d\lambda) = \frac{1}{|k|} \mathcal{M}_\rho(\xi^b, \eta^a; d\lambda). \quad (121)$$

In view of the compactness of G , all the unitary irreducible representations (UIR's) are finite dimensional and the PVM of $\xi^b H_b$ is concentrated on a set of $n = \dim D$ points $\{\mu_m\}_m$, $\mu_m = \xi^b m_b$ where m_b is an eigenvalue of H_b , $b = 1, \dots, r$, while $m = 1, \dots, n$:

$$\Pi(\xi^b)(d\lambda) = \sum_{\{m_b\}} P_{\{m_b\}} \delta(\lambda - \xi^b m_b) d\lambda, \quad (122)$$

and where a Gelfand-Zetlin basis has been chosen in such a way that the rank-one projector $P_{\{m_b\}}$ projects on the eigenspace of the eigenvalue $\xi^b m_b$, which is the same eigenspace of the eigenvalues m_b , for any b . Then we can define the tomogram of the state ρ , $W_\rho(\eta^a; m)$, with respect to the representation D of the group G :

$$\begin{aligned} \text{Tr}(\rho D(\tilde{g})) &= \sum_{\{m_b\}} \exp(i\xi^b m_b) \text{Tr} [\exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a) P_{\{m_b\}}] \quad (123) \\ &= \sum_{\{m_b\}} \exp(i\xi^b m_b) \text{Tr} [\rho \exp(i\eta^a E_a) P_{\{m_b\}} \exp(-i\eta^a E_a)] \\ &= \sum_{\{m_b\}} \exp(i\xi^b m_b) W_\rho(\eta^a; \{m_b\}). \end{aligned} \quad (124)$$

In other words, the tomogram $\{W_\rho(\eta^a; \{m_b\})\}$ is a stochastic vector:

$$\sum_{\{m_b\}} W_\rho(\eta^a; \{m_b\}) = 1. \quad (125)$$

The component $W_\rho(\eta^a; \{m_b\})$ is the joint probability that a measure of any H_b in the rotated state $\tilde{\rho} = \exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a)$ is m_b respectively:

$$\begin{aligned} \text{Tr}[\tilde{\rho} H_{b'}] &= \text{Tr} \left[\exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a) \sum_{\{m_b\}} m_{b'} P_{\{m_b\}} \right] \\ &= \sum_{\{m_b\}} m_{b'} W_\rho(\eta^a; \{m_b\}) \end{aligned} \quad (126)$$

We observe explicitly that the tomogram can be viewed equivalently as a measure of the rotated observable $\exp(i\eta^a E_a) H_{b'} \exp(-i\eta^a E_a)$ in the state ρ . In other words, out of the fiducial set of rank one projectors $P_{\{m_b\}}$'s, one gets a tomographic set of rotated rank-one projectors. Of course, the density state ρ can be reconstructed from its tomogram W_ρ . To this aim, we observe that from the orthogonality relations we get

$$d^{(D)} \int_G \varphi(\tilde{g})^* D(\tilde{g}) d\tilde{g} = d^{(D)} \int_G \text{Tr} [\rho D(\tilde{g})]^* D(\tilde{g}) d\tilde{g} = \rho, \quad (127)$$

where $d^{(D)}$, the formal dimension, is the dimension of D divided by the Haar volume of the group. That is, taking into account the reality of the tomograms,

$$d^{(D)} \int_G \sum_{\{m_b\}} \exp(-i\xi^b m_b) W_\rho(\eta^a; \{m_b\}) D(\tilde{g}) d\tilde{g} = \rho. \quad (128)$$

We observe that the above equation may be further detailed in particular cases.

For instance, in the $SU(2)$ case with $D = D^j$ of $2j+1$ dimensions. We preliminarily note that, in general, as $\tilde{g} = gtg^{-1}$ with $t \in T$ and $g \in G$, for any summable group function f :

$$\int_G f(\tilde{g}) \mu_G(d\tilde{g}) = \int_T \int_G f(gtg^{-1}) \mu_G(dg) \mu_T(dt) \quad (129)$$

where μ_G and μ_T are normalized invariant measure on G and T respectively.

Then, in the canonical basis of the eigenvectors $\{|m\rangle\}$, $m = -j, \dots, j$, of J_z we have:

$$\begin{aligned} \rho_{m_1 m_2} &= d^{(D)} \int_G \sum_{m=-j}^j \exp(-i\xi m) W_\rho(\eta^a; m) D_{m_1 m_2}(\tilde{g}) d\tilde{g} = \\ &= \frac{d^{(D)}}{2\pi} \int_0^{2\pi} d\xi \int_G \sum_{m, m'=-j}^j \exp[i\xi(m' - m)] W_\rho(g; m) (D(g) |m'\rangle \langle m'| D^\dagger(g))_{m_1 m_2} dg = \\ &= d^{(D)} \sum_{m=-j}^j \int_G W_\rho(g; m) (D(g) |m\rangle \langle m| D^\dagger(g))_{m_1 m_2} dg. \end{aligned} \quad (130)$$

The expression $D(g) |m\rangle \langle m| D^\dagger(g)$ is just the action of the group on $\mathcal{H} \otimes \mathcal{H}^*$, where \mathcal{H} is the carrier space of D and \mathcal{H}^* its dual, the carrier space of the transpose representation $D^T(g^{-1}) : D^T(g^{-1})(m, \cdot) = (m, D(g^{-1})\cdot) = \langle m| D^\dagger(g)$.

Now, for $SU(2)$, the representations $D(g)$ and its complex conjugate $D^*(g) = D^T(g^{-1})$ are equivalent for any j , so that we can use the contravariant basis ([49], sec. 41): $\langle m| \mapsto (-1)^{j-m} | -m \rangle$, in such a way that the group action on $\mathcal{H} \otimes \mathcal{H}^*$ is equivariant with the group action on $\mathcal{H} \otimes \mathcal{H}$. This allows to use the group action $D \otimes D$ on $\mathcal{H} \otimes \mathcal{H}$ and the addition theorem to decompose the product representation:

$$D^j \otimes D^j = \bigoplus_{J=0}^{2j} D^J. \quad (131)$$

Finally, the reconstruction formula for the matrix element $\rho_{m_1 m_2}$ reads

$$\begin{aligned} \rho_{m_1 m_2} &= \sum_{m=-j}^j d^{(D)} \int_G W_\rho(g; m) (D(g) |m\rangle \langle m| D^\dagger(g))_{m_1 m_2} dg = \\ &= \sum_{m=-j}^j \sum_{J=0}^{2j} \sum_{M, M'=-J}^J (-1)^{2j-m-m_2} d^{(D)} \int_G W_\rho(g; m) \langle m_1 | \langle -m_2 | JM \rangle \end{aligned}$$

$$\begin{aligned}
 & \times \langle JM | D^J(g) | JM' \rangle \langle JM' | m \rangle | -m \rangle dg = \\
 & = \sum_{m=-j}^j \sum_{J=0}^{2j} \sum_{M=-J}^J (-1)^{2j-M-m-m_2} (2J+1) \begin{bmatrix} j & j & J \\ m_1 & -m_2 & -M \end{bmatrix} \times \\
 & \times \begin{bmatrix} j & j & J \\ m & -m & 0 \end{bmatrix} d^{(D)} \int_G W_\rho(g; m) D_{M0}^J(g) dg. \tag{132}
 \end{aligned}$$

where the Wigner $3j$ -symbols are introduced.

The above equation may be related to the reconstruction formulae contained in [50].

In fact, by observing that

$$\begin{aligned}
 W_\rho(g; m) &= W_\rho^*(g; m) = \sum_{m'_1, m'_2=-j}^j \rho_{m'_1 m'_2}^* (D(g) | m \rangle \langle m | D^\dagger(g))^*_{m'_2 m'_1} = \\
 &= \sum_{m'_1, m'_2=-j}^j \rho_{m'_1 m'_2}^* \sum_{J'=0}^{2j} \sum_{M'=-J'}^{J'} \left[(-1)^{2j-M'-m-m'_1} D_{M'0}^{J'}(g) \right]^* \times \\
 & \times (2J+1) \begin{bmatrix} j & j & J' \\ m'_2 & -m'_1 & -M' \end{bmatrix} \begin{bmatrix} j & j & J' \\ m & -m & 0 \end{bmatrix}, \tag{133}
 \end{aligned}$$

the integration over the group yields $\delta_{J,J'} \delta_{M,M'}$. By means of the well known identities

$$\begin{aligned}
 (2J+1) \sum_{m=-j}^j \begin{bmatrix} j & j & J \\ m & -m & 0 \end{bmatrix} \begin{bmatrix} j & j & J \\ m & -m & 0 \end{bmatrix} &= 1, \tag{134} \\
 \sum_{J=0}^{2j} \sum_{M=-J}^J (2J+1) \begin{bmatrix} j & j & J \\ m'_2 & -m'_1 & -M \end{bmatrix} \begin{bmatrix} j & j & J \\ m_1 & -m_2 & -M \end{bmatrix} &= \delta_{m_1, m'_2} \delta_{m_2, m'_1},
 \end{aligned}$$

the l.h.s. of eq. (132) eventually gives $\rho_{m_2 m_1}^* = \rho_{m_1 m_2}$.

In the general $U(n)$ case, when the used representation and its conjugate are equivalent, one can try to follow the previous route to perform the reconstruction.

However, we note that $SU(2)$ can be embedded irreducibly in the defining representation of $U(n)$, for any n . So, the above reconstruction formula is general and can be used to reconstruct density states out of the restriction of the $U(n)$ tomograms to the subgroup $SU(2)$.

Back to the general analysis, we remark that as $\varphi(\tilde{g}) = \text{Tr}[\rho D(\tilde{g})]$ is a function on the group G of positive type, the theorem of Naimark [39] states that there exist a unitary representation U on a Hilbert space determined by a GNS construction and a cyclic vector ψ_0 such that

$$\varphi(g) = (\psi_0, U(g) \psi_0). \tag{135}$$

As a result, following a procedure similar to that discussed in sec.6, if ρ is a pure state $|\psi\rangle\langle\psi|$, then U and ψ_0 are unitarily equivalent to D and ψ respectively. When ρ

is a mixed state of rank r , then U is reducible, and can be put in block form of r blocks \mathcal{V} unitarily equivalent to $D : \mathcal{V}(g) = VD(g)V^\dagger$. Then ρ can be reconstructed by

$$d^{(U)} \int_G \varphi^*(g) \mathcal{V}(g) dg = V \rho V^\dagger. \quad (136)$$

This extends Proposition 3 of Sec. 6 to the compact group case.

Also Proposition 2 of Sec. 6 can be extended to the present case. However, we remark that when an arbitrary irreducible representation has been chosen instead of the defining one, the Cartan subalgebra operators are not a complete set any further, and a Gelfand-Zetlin [44, 45] basis has to be determined by considering a set of Casimir operators of subgroups: for instance, in the $SU(3)$ case, the isotopic spin operator. In the next section, we present a discussion of Gelfand-Zetlin basis construction making contact with tomographic representations. In fact, tomograms depend not only on the group parameters, playing the role of “positions” in configuration space, but also on Gelfand-Zetlin basis labels, that play the role of “conjugate momenta”.

9. Gelfand-Zetlin bases

Let us comment first on how the tomograms constructed using a unitary representation of a group G are connected not only with the group itself but also with the choice of the chain of the subgroups of the group which is used to determine the basis in the Hilbert space on which is acting the irreducible representation of the group. In fact, the tomogram $W^\alpha(g, m)$ is a function of the group element g , of the Casimir label of the representation α and of the collective label m which determines the basis vector in the corresponding Hilbert space. We remind how this label m is determined. For example for the $SU(2)$ -group the natural choice of the parameter m is the spin projection on z -axis for a given value $\alpha = J$ of the Casimir operator \mathbf{J}^2 .

In a purely group-theoretical formalism that does not use any “physical” interpretation of the index m (and index α as angular momentum J) the basis is determined by the Lie algebra generator J_z of the subgroup $U(1)$ of the group $SU(2)$: one has the chain $SU(2) \supset U(1)$. In the case of $SU(3)$ the Gelfand-Zetlin basis is determined by the chain $SU(3) \supset SU(2) \supset U(1)$ of subgroups embedded into $SU(3)$. In fact one determines the basis using first the Casimir operators of $SU(3)$, then using the Casimir operator of $SU(2)$ (corresponding to the value of the isotopic spin \mathbf{T}^2) and the generators of the Cartan subalgebra providing the weights m_1, m_2 . Due to multiplicity of the weights, to label Cartan generators eigenvectors one needs the Casimir operator \mathbf{T}^2 of the subgroup $SU(2)$ embedded into the initial group $SU(3)$. For any higher group $SU(n)$, the Gelfand-Zetlin basis is constructed by using the chain $SU(n) \supset SU(n-1) \supset \dots \supset U(1)$ of embedded subgroups.

But there exist other possibilities to use different chains of subgroups embedded into the initial group G . For example one can construct the basis for the irreducible representations of the group $SU(6)$ by using the subgroup $SU(3) \otimes SU(2)$ embedded

into $SU(6)$. The basis obtained in this way provides the possibility to get "quantum numbers" corresponding to standard spins (i.e., associated with the group $SU(2)$) and the charges associated with the group $SU(3)$. In fact, the ambiguity in the choice of the subgroup chains determining the basis index m corresponds to the ambiguity in the choice of the complete set of commuting observables, operators acting on the Hilbert space of the irreducible representation of the group G . Of course, the basis vectors $\{|\alpha, m\rangle\}$ and $\{|\alpha, m'\rangle\}$ determined by Casimirs α and quantum numbers m and m' , associated with two different chains of subgroups embedded into the group G , or with two different complete sets of commuting observables, are related by a unitary transformation $U : |\alpha, m\rangle = U |\alpha, m'\rangle$. In terms of the corresponding rank-one projectors this reads: $P_{\{\alpha, m\}} = U P_{\{\alpha, m'\}} U^\dagger$.

From the tomographic point of view, the role played by U in relating tomograms associated with different chains of embedded subgroups is the following. Recalling the definition of tomogram of the density state ρ , we obtain:

$$W_\rho^\alpha(g, m) = W_{U^\dagger \rho U}^\alpha(g, m'). \quad (137)$$

In other words, the tomogram of the density state ρ in the basis $\{|\alpha, m\rangle\}$ with respect to the representation D^α is just the tomogram of the transformed density state $U^\dagger \rho U$ in the transformed basis $\{|\alpha, m'\rangle = U^\dagger |\alpha, m\rangle\}$ with respect to the transformed representation $U^\dagger D^\alpha U$.

10. The paradigmatic case of $SU(3)$

We illustrate the previous analysis by considering the paradigmatic example of the group $SU(3)$.

The basis vector of irreducible representations of $SU(3)$ are labelled by the eigenvalues C_1 and C_2 of the Casimir operators \hat{C}_1 and \hat{C}_2 . These in the case of the $SU(2)$ group reduce to the spin Casimir operator \mathbf{J}^2 with eigenvalues $j(j+1)$, $j = 0, 1/2, \dots$. After fixing the representation $D^{(C_1, C_2)}$ by a pair (C_1, C_2) , there is a Gelfand-Zetlin basis $\{|m_1, m_2; m_3\rangle\}$ of the Hilbert space acted upon by $D^{(C_1, C_2)}$, labelled by three quantum numbers m_1, m_2 and m_3 .

The quantum numbers m_1, m_2 are the spectra of the Cartan subalgebra $\{H_a\}$ operators, i.e.

$$H_a |m_1, m_2; m_3\rangle = m_a |m_1, m_2; m_3\rangle, \quad a = 1, 2 \quad (138)$$

and m_3 is eigenvalue of the Casimir operator \mathbf{T}^2 associated with the $SU(2)$ (isotopic spin) subgroup of $SU(3)$

$$\mathbf{T}^2 |m_1, m_2, m_3\rangle = m_3(m_3 + 1) |m_1, m_2, m_3\rangle \quad (139)$$

Let us rotate the basis $|m_1, m_2; m_3\rangle$ by applying the representation matrix $D(g)$ of $SU(3)$. We get a new basis

$$|m_1, m_2; m_3; g\rangle := D(g) |m_1, m_2; m_3\rangle. \quad (140)$$

Then we consider for a group element \tilde{g} the mean value of $D(\tilde{g})$ in the density state ρ belonging to the Hilbert space of the irreducible representation; in other words we get the Naimark positive function

$$\varphi^{(C_1, C_2)}(\tilde{g}) = \text{Tr}[\rho D^{(C_1, C_2)}(\tilde{g})]. \quad (141)$$

Here, dropping the label (C_1, C_2) ,

$$D(\tilde{g}) = D(g)D(t)D^\dagger(g), \quad D(t) = \exp[i(\xi^1 H_1 + \xi^2 H_2)]. \quad (142)$$

Consider the standard spectral decomposition of the unitary matrix $D(\tilde{g})$:

$$\begin{aligned} D(\tilde{g}) &= \sum_{m_1, m_2, m_3} e^{i(\xi^1 m_1 + \xi^2 m_2)} |m_1, m_2, m_3; g\rangle \langle m_1, m_2, m_3; g| \\ &= \sum_{m_1, m_2, m_3} e^{i(\xi^1 m_1 + \xi^2 m_2)} P_{m_1, m_2, m_3}(g), \end{aligned} \quad (143)$$

where $P_{m_1, m_2, m_3}(g)$ is the rank-one projector corresponding to $|m_1, m_2, m_3; g\rangle$. In other words $\{P_{m_1, m_2, m_3}(g)\}$ is the PVM of the observable $\xi^1 H_1 + \xi^2 H_2$, which is a concentrated measure on the points $\{\xi^1 m_1 + \xi^2 m_2\} \subset \mathbb{R}$. We get an expression for the positive function in the form

$$\begin{aligned} \varphi_\rho(\tilde{g}) &= \sum_{m_1, m_2, m_3} e^{i(\xi^1 m_1 + \xi^2 m_2)} \text{Tr}[\rho P_{m_1, m_2, m_3}(g)] \\ &=: \sum_{m_1, m_2, m_3} e^{i(\xi^1 m_1 + \xi^2 m_2)} W_\rho(m_1, m_2, m_3; g) \end{aligned} \quad (144)$$

Here we have defined the tomogram of ρ in the irreducible representation $D^{(C_1, C_2)}$ of $SU(3)$ as the function $W_\rho(m_1, m_2, m_3; g) = \text{Tr}[\rho P_{m_1, m_2, m_3}(g)]$.

11. Inverse tomographic problem

Consider an operator A , acting on the same Hilbert space of the irreducible representation D^α of the finite group G_K . Using the tomographic symbols $\left\{ \left(V_{g_j}^{\alpha\dagger} A V_{g_j}^\alpha \right)_{mm} \right\}_{j=1}^K$ of the operator A , the formula holds:

$$\frac{n_\alpha}{K} \sum_{j=1}^K \sum_{m=1}^{n_\alpha} e^{-i\theta_m^\alpha(g_j)} \left(V_{g_j}^{\alpha\dagger} A V_{g_j}^\alpha \right)_{mm}^* D_{rs}^\alpha(g_j) \quad (145)$$

$$\begin{aligned} &= \frac{n_\alpha}{K} \sum_{j=1}^K \text{Tr}[A D^\alpha(g_j)]^* D_{rs}^\alpha(g_j) \\ &= \sum_{p,q} A_{pq}^* \frac{n_\alpha}{K} \sum_{j=1}^K D^\alpha(g_j)_{qp}^* D_{rs}^\alpha(g_j) = A_{sr}^* = A_{rs}^\dagger \end{aligned} \quad (146)$$

When the operator A is an observable, i.e., $A^\dagger = A$, the equation above is a reconstruction formula for A .

Let us consider the family of n_α -dimensional vectors

$$\{v_m^\alpha(g_j)\} = \left\{ \left(V_{g_j}^{\alpha\dagger} A V_{g_j}^\alpha \right)_{mm} \right\}, j = 1, \dots, K. \quad (147)$$

In view of the above formula, as $\left(V_{g_j}^{\alpha\dagger} A V_{g_j}^{\alpha}\right)_{mm}^* = \left(V_{g_j}^{\alpha\dagger} A V_{g_j}^{\alpha}\right)_{mm}$, they satisfy the self-consistency relation written in terms of a reproducing kernel $R_{pm}^{\alpha}(g_j, g_h)$:

$$\frac{n_{\alpha}}{K} \sum_{j=1}^K \sum_{m=1}^{n_{\alpha}} R_{pm}^{\alpha}(g_j, g_h) v_m^{\alpha*}(g_j) = v_p^{\alpha*}(g_h), \quad (148)$$

$$R_{pm}^{\alpha}(g_j, g_h) := e^{-i\theta_m^{\alpha}(g_j)} \sum_{r,s=1}^{n_{\alpha}} \left(V_{g_h}^{\alpha\dagger}\right)_{pr} D_{rs}^{\alpha}(g_j) \left(V_{g_h}^{\alpha}\right)_{sp}.$$

The vectors $\mathbf{v}^{\alpha}(g_j)$ can be chosen as stochastic vectors only if A is a positive semidefinite observable: i.e., a density state ρ , after normalization.

In fact, after diagonalization $A = U \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_{n_{\alpha}}] U^{\dagger}$, we may choose the arbitrary diagonalizing matrix V_e associated to the neutral element e of the group to be $U, V_e = U$. If A is diagonal we choose the identity matrix as V_e . In this way, we get as corresponding column vector just $(\lambda_1, \lambda_2, \dots, \lambda_{n_{\alpha}})^T$, which is a (normalizable) stochastic vector only when all the eigenvalues are nonnegative.

However, the above condition is by no means sufficient: a family of stochastic vectors can be associated to any observable A .

For instance, in the triangle group case, consider the tomographic symbols of the observable A :

$$A = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} U^{\dagger} = \begin{bmatrix} \lambda_1 \cos^2 \frac{\theta}{2} - \lambda_2 \sin^2 \frac{\theta}{2} & -\frac{\lambda_1 + \lambda_2}{2} \sin \theta e^{i\phi} \\ -\frac{\lambda_1 + \lambda_2}{2} \sin \theta e^{-i\phi} & \lambda_1 \sin^2 \frac{\theta}{2} - \lambda_2 \cos^2 \frac{\theta}{2} \end{bmatrix}, \quad (149)$$

where

$$\lambda_1, \lambda_2 > 0, \quad U = \begin{bmatrix} \cos \frac{\theta}{2} e^{i\frac{\phi+\psi}{2}} & \sin \frac{\theta}{2} e^{i\frac{\phi-\psi}{2}} \\ -\sin \frac{\theta}{2} e^{-i\frac{\phi-\psi}{2}} & \cos \frac{\theta}{2} e^{-i\frac{\phi+\psi}{2}} \end{bmatrix}. \quad (150)$$

The tomographic symbols $\left\{ \left(V_{g_j}^{\alpha\dagger} A V_{g_j}^{\alpha}\right)_{mm} \right\}_{j=1}^K$ of A are in consequence

$$\left\{ \begin{bmatrix} \lambda_1 \cos^2 \frac{\theta}{2} - \lambda_2 \sin^2 \frac{\theta}{2} \\ \lambda_1 \sin^2 \frac{\theta}{2} - \lambda_2 \cos^2 \frac{\theta}{2} \end{bmatrix} \right\}_{j=1,2,3},$$

$$\left\{ \frac{1}{2} \begin{bmatrix} \lambda_1 - \lambda_2 + (\lambda_1 + \lambda_2) \cos(\phi + \alpha) \sin \theta \\ \lambda_1 - \lambda_2 - (\lambda_1 + \lambda_2) \cos(\phi + \alpha) \sin \theta \end{bmatrix} \right\}_{\alpha=0, \frac{2\pi}{3}, \frac{4\pi}{3}} \quad (151)$$

Picking $\theta = \pi/2$, $\lambda_1 - \lambda_2 = 1$, we have the vectors

$$\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 1 + (1 + 2\lambda_2) \cos(\phi + \alpha) \\ 1 - (1 + 2\lambda_2) \cos(\phi + \alpha) \end{bmatrix}_{\alpha=0, \frac{2\pi}{3}, \frac{4\pi}{3}} \quad (152)$$

so that, putting $M(\phi) = \max\{|\cos(\phi + \alpha)|, \alpha = 0, 2\pi/3, 4\pi/3\}$, we get the stochasticity condition

$$(1 + 2\lambda_2) M(\phi) \leq 1 \Leftrightarrow 0 \leq \lambda_2 \leq \frac{1}{2M(\phi)} - \frac{1}{2} \quad (153)$$

which gives a nonzero λ_2 for $M(\phi) < 1$, that is, for $0 < \phi < \pi/3$.

So, we have to address the problem of stating a sufficient, and necessary, condition for an assigned family of n -dimensional stochastic vectors $\{\boldsymbol{\tau}(g_j)\}_{j=1}^K$ to be the tomogram of a state with respect to a given n -dimensional irreducible representation D^α of the group G_K :

$$\exists \rho : \tau_m(g_j) = \left(V_{g_j}^{\alpha\dagger} \rho V_{g_j}^\alpha \right)_{mm}, \quad m = 1, \dots, n, \quad j = 1, \dots, K. \quad (154)$$

A sufficient (and necessary) condition can be stated in terms of positivity of a suitable group function.

As the diagonal matrices d_g^α 's depend only on representation D^α and are supposed to be known, we can define a normalized group function ψ^α as

$$\psi^\alpha(g_j) = \sum_{m=1}^n e^{i\theta_m^\alpha(g_j)} \tau_m(g_j). \quad (155)$$

Requiring that the tomographic symbols $\{v_m^\alpha(g_j)\}$ of the operator

$$\frac{n_\alpha}{K} \sum_{j=1}^K \sum_{m=1}^n e^{-i\theta_m^\alpha(g_j)} \tau_m(g_j) D^\alpha(g_j) \quad (156)$$

constructed using this group function are just the assigned stochastic vectors, the self-consistency relation (148) yields.

$$\frac{n_\alpha}{K} \sum_{j=1}^K \sum_{m=1}^n R_{pm}^\alpha(g_j, g_h) \tau_m(g_j) = v_p^\alpha(g_h) = \tau_p(g_h). \quad (157)$$

This is a necessary condition that the stochastic vectors must satisfy in order to solve the posed problem, we may call it a condition of compatibility of the τ 's with the representation D^α .

Besides, requiring that the operator (156) is self-adjoint gives

$$\frac{n_\alpha}{K} \sum_{j=1}^K \sum_{m=1}^n e^{-i\theta_m^\alpha(g_j)} \tau_m(g_j) D^\alpha(g_j) = \frac{n_\alpha}{K} \sum_{j=1}^K \sum_{m=1}^n e^{-i\theta_m^\alpha(g_j^{-1})} \tau_m(g_j) D^\alpha(g_j^{-1}). \quad (158)$$

Finally, we check whether ψ^α is a positive-type function. If the answer to the check is in the affirmative, the observable (156) is just a density state ρ_τ^α

$$\rho_\tau^\alpha := \frac{n}{K} \sum_{j=1}^K (\psi^\alpha(g_j))^* D^\alpha(g_j), \quad (159)$$

such that its tomogram with respect to D^α is just the assigned family of stochastic vectors:

$$\{\mathbf{W}^\alpha(g_j)\}_{j=1}^K = \{\boldsymbol{\tau}(g_j)\}_{j=1}^K. \quad (160)$$

In this case, we call tomogram the given family. Equivalently, we can write:

$$\psi^\alpha(g_j) = \text{Tr} [\rho_\tau^\alpha D^\alpha(g)]. \quad (161)$$

So, the positivity condition implies that the stochastic family is compatible with D^α and this, in turn, implies that, in the decomposition of a group function with respect

to the matrix elements of all the irreducible representations, the normalized function ψ^α has only components in the representation D^α . This completes the proof.

Example. We now illustrate the above analysis with the example of the D^2 representation of the triangle group.

The more general stochastic 2-dimensional distribution on the group reads

$$\tau(g_j) = \frac{1}{2} \begin{bmatrix} 1 + x_j \\ 1 - x_j \end{bmatrix}, \quad -1 \leq x_j \leq 1, \quad j = 1, 2, \dots, 6. \quad (162)$$

The compatibility condition with D^2 yields

$$x_4 + x_5 + x_6 = 0. \quad (163)$$

The Hermiticity condition gives

$$x_2 = x_3. \quad (164)$$

Construct the group function ψ^2 using eq.(155) and the above self-consistency and Hermiticity relations. The Naimark matrix $\psi^2(g_i^{-1}g_j)$, $i, j = 1, \dots, 6$, has the following distinct eigenvalues:

$$0, \quad \frac{3}{2} \pm \frac{1}{2} \sqrt{3(3x_2^2 + 4x_5x_6 + 4x_5^2 + 4x_6^2)}. \quad (165)$$

Positivity requires that

$$3x_2^2 + 4x_5x_6 + 4x_5^2 + 4x_6^2 \leq 3 \quad (166)$$

This constraint can be easily understood after diagonalization, putting:

$$x_5 + x_6 = x, \quad x_5 - x_6 = \sqrt{3}y, \quad x_2 = z, \quad (167)$$

which yields

$$x^2 + y^2 + z^2 = r^2 \leq 1, \quad (168)$$

allowing the identification with the condition satisfied by density states in two dimensions, discussed in subsec.7.2. In other words, there exists a one-to-one correspondence between density states and stochastic distributions satisfying positivity condition, which result just their tomograms.

In conclusion, in the space of parameters $\{-1 \leq x_j \leq 1\}$, $j = 1, \dots, 6$, the relations $\{x_2 = x_3, \quad x_4 = -x_5 - x_6\}$ define the set of stochastic vectors in one to one correspondence with tomographic symbols of observables in the representation D^2 , which contains the unit ball of density states defined by the constraint $3x_2^2 + 4x_5x_6 + 4x_5^2 + 4x_6^2 \leq 3$.

Now, to conclude this example, choose a tomographic family of stochastic vectors by means of a suitable point (x, y, z) , corresponding to the density state

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + z & x - iy \\ x + iy & 1 - z \end{pmatrix} \quad (169)$$

which is diagonalized, when $x + iy \neq 0$, by the unitary matrix u

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} (z - r)(r^2 - rz)^{-\frac{1}{2}} & (z + r)(r^2 + rz)^{-\frac{1}{2}} \\ (x + iy)(r^2 - rz)^{-\frac{1}{2}} & (x + iy)(r^2 + rz)^{-\frac{1}{2}} \end{pmatrix} \quad (170)$$

The matrix u corresponding to the diagonal case $x + iy = 0$ cannot be obtained by a limit procedure.

In view of the Naimark theorem and construction of sec.6, it is possible to exhibit explicit formulae for a unitary representation and a pure cyclic vector state ξ to represent canonically the ψ^2 corresponding to the chosen point (x, y, z) .

One gets a four dimensional Hilbert space, acted upon by the following reducible representation of the group S_3

$$\begin{pmatrix} D^2 & 0 \\ 0 & D^2 \end{pmatrix} \quad (171)$$

and the following density matrix for the pure cyclic state

$$\rho_\xi = U \begin{pmatrix} \rho_- & 0 & 0 & \sqrt{\rho_- \rho_+} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\rho_- \rho_+} & 0 & 0 & \rho_+ \end{pmatrix} U^\dagger. \quad (172)$$

Here $\rho_\mp = \frac{1}{2}(1 \mp r)$ are the eigenvalues of ρ and U is a 4×4 - matrix in block-form

$$U = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}. \quad (173)$$

One can check that the Hermitian matrix ρ_ξ has trace one and $\text{Tr} [\rho_\xi^2] = 1$ so it is the density matrix of a pure state ξ . Since U and ρ are explicitly given in terms of the stochastic distribution $\tau(x, y, z)$, we got the relation between tomographic probability distributions on the group and Naimark pure cyclic vector states ξ .

Back to the general finite group case, suppose there are two (or more) irreducible different representation D^α, D^β with the same dimensionality $n_{\alpha\beta}$, and that the stochastic vectors $\{\tau^\alpha(g_j)\}_{j=1}^K$, corresponding to a state ρ_τ^α , make positive the function:

$$\psi^\alpha(g_j) = \sum_{m=1}^n e^{i\theta_m^\alpha(g_j)} \tau_m^\alpha(g_j). \quad (174)$$

We can construct the set of stochastic vectors $\{\tau^\beta(g_j)\}_{j=1}^K$ corresponding to the same state ρ_τ^α and making positive the function:

$$\psi^\beta(g_j) = \sum_{m=1}^n e^{i\theta_m^\beta(g_j)} \tau_m^\beta(g_j) = \text{Tr} [\rho_\tau^\alpha D^\beta(g)]. \quad (175)$$

In fact, in view of the reconstruction formula (146), we have:

$$\begin{aligned} & \frac{n_\alpha}{K} \sum_{j=1}^K \sum_{m=1}^{n_\alpha} e^{-i\theta_m^\alpha(g_j)} \tau_m^\alpha(g_j) \sum_{r,s=1}^{n_\alpha} (V_{gh}^{\beta\dagger})_{pr} D_{rs}^\alpha(g_j) (V_{gh}^\beta)_{sp} \\ &= \sum_{r,s=1}^{n_\alpha} (V_{gh}^{\beta\dagger})_{pr} \rho_{rs} (V_{gh}^\beta)_{sp} = \tau_p^\beta(g_h). \end{aligned} \quad (176)$$

Now, it can happen that the same family of stochastic vectors satisfies the positivity condition of both the group functions ψ^α, ψ^β . Then, in view of eq.(160), the tomograms $\mathbf{W}^\alpha, \mathbf{W}^\beta$ are the same and two possibilities can present: or $V^\alpha = V^\beta$ either $V^\alpha \neq V^\beta$ for any group element g_j .

In the first case, in view of eq.(159), the reconstructed density states the same: $\rho_\tau^\alpha = \rho_\tau^\beta$. For example, this is the case of the two inequivalent 2-dimensional representations of the tetrahedron group, related to the D^2 representation of the triangle group S_3 as $D^\alpha = \{D^2, D^2\}$ and $D^\beta = \{D^2, -D^2\}$.

In the second case, the states are different: $\rho_\tau^\alpha \neq \rho_\tau^\beta$. This is the case, for instance, of the 3-dimensional irreducible representations of $SU(3)$, $D^\alpha = D, D^\beta = D^*$, where $\rho_\tau^\beta = \rho_\tau^{\alpha*} \neq \rho_\tau^\alpha$. This result is obtained by a straightforward and obvious generalization of all the above formulae and conditions to the case of compact groups.

Briefly, given on the group G an irreducible representation D and the stochastic vector function $\{\tau_{\{m_b; m_c\}}(g)\}$, whose components are labelled by using a suitable Gelfand-Zetlin basis, one can construct the group function

$$\psi(\tilde{g}) = \sum_{\{m_b; m_c\}} \exp(-i\xi^b m_b) \tau_{\{m_b; m_c\}}(g). \quad (177)$$

By using eq. (127), a density state ρ can be recovered by $\psi(\tilde{g})$ iff this function is of positive-type. Moreover, if the stochastic vector function is compatible with D , it is the tomogram of ρ :

$$W_\rho(g; \{m_b; m_c\}) = \tau_{\{m_b; m_c\}}(g), \quad (178)$$

and this solves completely the inverse tomographic problem.

Compatibility condition may be written as

$$\begin{aligned} \tau_{\{m'_b; m'_c\}}(g') &= d^{(D)} \int_G \psi(\tilde{g})^* (D^\dagger(g') D(\tilde{g}) D(g'))_{\{m'_b; m'_c\} \{m'_b; m'_c\}} d\tilde{g} \\ &= d^{(D)} \int_G \sum_{\{m_b; m_c\}} e^{-i\xi^b m_b} \tau_{\{m_b; m_c\}}(g) (D^\dagger(g') D(\tilde{g}) D(g'))_{\{m'_b; m'_c\} \{m'_b; m'_c\}} d\tilde{g} \end{aligned} \quad (179)$$

where $\tilde{g} = g \exp(\xi^b H_b) g^{-1}$ and H_b 's are the generator of the Cartan subalgebra, as usual.

We remark that checking the positivity of a compact group function like the above $\psi(\tilde{g})$ amounts to an infinite number of operations.

However, if an irreducible representation $D(G_K)$ of a finite group can be found in $D(G)$, then one can limit to check the positivity condition on the finite group only for one $K \times K$ -matrix. Assume that this holds true. For example, this is the case of the defining representation of $U(2)$, which contains the representation D^2 of the group S_3 .

Besides, suppose that $\psi(\tilde{g})$ satisfies the compatibility condition with D , so that it has no components with respect to other irreducible representations. In this situation the positivity of ψ on G can be checked on G_K .

In fact, if ψ is positive on G_K , we get a density state ρ on the n -dimensional Hilbert space on which D acts such that

$$\psi(g_j) = \text{Tr}[\rho D(g_j)] = \sum_{r,s=1}^n \rho_{sr} D_{rs}(g_j). \quad (180)$$

By hypothesis ψ can be expanded using only the matrix elements of D :

$$\psi(g) = \sum_{r,s=1}^n c_{rs} D_{rs}(g), \quad (181)$$

that are orthogonal on G as well on G_K

$$\delta_{r,q} \delta_{s,p} = \frac{n}{K} \sum_{j=1}^K D_{rs}^*(g_j) D_{qp}(g_j) = d^{(D)} \int_G D_{rs}^*(g) D_{qp}(g) dg. \quad (182)$$

It readily follows that

$$c_{rs} = \rho_{sr} \Rightarrow \psi(g) = \text{Tr}[\rho D(g)] \quad (183)$$

and $\psi(g)$ is positive on G .

12. Conclusions

To conclude, we summarize the main results of our work. For states of finite dimensional C^* -algebras we have introduced the notion of tomographic probability distribution. This concept provides the possibility of clarifying new aspects of C^* -algebras related to information characteristics of the probability distributions like different kinds of entropies.

These tomograms were also introduced for finite and compact groups by using known unitary finite dimensional irreducible representations of these groups. The tomographic probability vectors (tomograms) introduced for those groups were shown to contain complete information on the quantum states (Hermitian, trace-class, nonnegative matrices) associated with the irreducible unitary representations of those groups.

The notion of Naimark matrix and its properties were used to study necessary and sufficient conditions for the stochastic vectors defined on the finite or compact groups to be tomographic probability distributions. The Naimark theorem on positive-type group functions was shown to play a key role in the problem of connecting the tomographic probability vectors on the group with the density states on the Hilbert space of the irreducible representations of the group.

The paradigmatic examples of two groups, the group S_3 of permutations of three points and $SU(3)$, were discussed in detail. The general construction of the $U(n)$ group (and other classical groups) tomograms was presented by using the Gelfand-Zetlin basis labels of the tomographic probability vectors.

The notion of tomographic probabilities introduced for finite C^* -algebras was, in fact, shown to coincide with that of tomographic probability vectors associated to finite

unitary groups. The probability vectors defined on finite or compact groups establish a relation between the group structure and the structure of the simplexes containing those probability vectors. An analogous relation exists between finite C^* –algebras and those simplexes, thanks to the existence of tomographic probability vectors defined on the C^* –algebras.

Thus, for finite and compact groups, their group algebras and abstract C^* –algebras were considered in the unifying framework of the tomographic approach, where the tomograms provide the possibility to describe completely all the kinds of quantum states, both pure and mixed ones.

For example, the spin states (qu-dits) associated with $SU(2)$ –group irreducible representations can be alternatively described by the spin-tomographic probability distributions of measurable spin-projections on the quantization axes.

We will develop these aspects of the tomographic approach to systems with the discussed finite or compact symmetry groups in future papers.

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